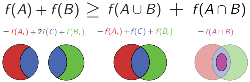
Submodular Functions, Optimization, and Applications to Machine Learning
— Spring Quarter, Lecture 12 —

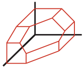
http://www.ee.washington.edu/people/faculty/bilmes/classes/ee563_spring_2018/

Prof. Jeff Bilmes

University of Washington, Seattle
Department of Electrical Engineering
http://melodi.ee.washington.edu/~bilmes

May 7th, 2018





Cumulative Outstanding Reading

- Read chapter 1 from Fujishige's book.
- Read chapter 2 from Fujishige's book.
- Read chapter 3 from Fujishige's book.
- Read chapter 4 from Fujishige's book.

Announcements, Assignments, and Reminders

- Next homework is posted on canvas. Due Thursday 5/10, 11:59pm.
- As always, if you have any questions about anything, please ask then via our discussion board (`https://canvas.uw.edu/courses/1216339/discussion_topics`). Can meet at odd hours via zoom (send message on canvas to schedule time to chat).

Class Road Map - EE563

- L1(3/26): Motivation, Applications, & Basic Definitions,
- L2(3/28): Machine Learning Apps (diversity, complexity, parameter, learning target, surrogate).
- L3(4/2): Info theory exs, more apps, definitions, graph/combinatorial examples
- L4(4/4): Graph and Combinatorial Examples, Matrix Rank, Examples and Properties, visualizations
- L5(4/9): More Examples/Properties/ Other Submodular Defs., Independence,
- L6(4/11): Matroids, Matroid Examples, Matroid Rank, Partition/Laminar Matroids
- L7(4/16): Laminar Matroids, System of Distinct Reps, Transversals, Transversal Matroid, Matroid Representation, Dual Matroids
- L8(4/18): Dual Matroids, Other Matroid Properties, Combinatorial Geometries, Matroids and Greedy.
- L9(4/23): Polyhedra, Matroid Polytopes, Matroids → Polymatroids
- L10(4/29): Matroids → Polymatroids, Polymatroids, Polymatroids and Greedy,
- L11(4/30): Polymatroids, Polymatroids and Greedy
- L12(5/2):
- L13(5/7):
- L14(5/9):
- L15(5/14):
- L16(5/16):
- L17(5/21):
- L18(5/23):
- L–(5/28): Memorial Day (holiday)
- L19(5/30):
- L21(6/4): Final Presentations maximization.

Last day of instruction, June 1st. Finals Week: June 2-8, 2018.

Vector rank, rank(x), is submodular

- Recall that the matroid rank function is submodular.
- The vector rank function rank(x) also satisfies a form of submodularity, namely one defined on the real lattice.

Theorem 12.2.1 (vector rank and submodularity)

Let P be a polymatroid polytope. The vector rank function rank : $\mathbb{R}_+^E \to \mathbb{R}$ with rank$(x) = \max(y(E) : y \leq x, y \in P)$ satisfies, for all $u, v \in \mathbb{R}_+^E$

$$\text{rank}(u) + \text{rank}(v) \geq \text{rank}(u \vee v) + \text{rank}(u \wedge v) \tag{12.1}$$

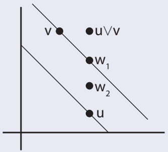
More on polymatroids

Theorem 12.2.1

A polymatroid can equivalently be defined as a pair (E, P) where E is a finite ground set and $P \subseteq R_+^E$ is a compact non-empty set of independent vectors such that

1. *every subvector of an independent vector is independent (if $x \in P$ and $y \leq x$ then $y \in P$, i.e., down closed)*
2. *If $u, v \in P$ (i.e., are independent) and $u(E) < v(E)$, then there exists a vector $w \in P$ such that*

$$u < w \leq u \vee v \qquad (12.20)$$



Corollary 12.2.2

The independent vectors of a polymatroid form a convex polyhedron in $\mathbb{R}_+^E$.

More on polymatroids

For any compact set P, b is a base of P if it is a maximal subvector within P. Recall the bases of matroids. In fact, we can define a polymatroid via vector bases (analogous to how a matroid can be defined via matroid bases).

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2. *if b, c are bases of P and d is such that $b \wedge c < d < b$, then there exists an f, with $d \wedge c < f \leq c$ such that $d \vee f$ is a base of P*
3. *All of the bases of P have the same rank.*

Note, all three of the above are required for a polymatroid (a matroid analogy would require the equivalent of only the first two).

Matroid instance of Theorem ??

- Considering Theorem **??**, the matroid case is now a special case, where we have that:

Corollary 12.2.2

We have that:

$$\max\{y(E) : y \in P_{\textit{ind. set}}(M), y \leq x\} = \min\{r_M(A) + x(E \setminus A) : A \subseteq E\} \tag{12.21}$$

where r_M is the matroid rank function of some matroid.

Polymatroidal polyhedron and greedy

- Let $(E, \mathcal{I})$ be a set system and $w \in \mathbb{R}_+^E$ be a weight vector.
- Recall greedy algorithm: Set $A = \emptyset$, and repeatedly choose $y \in E \setminus A$ such that $A \cup \{y\} \in \mathcal{I}$ with $w(y)$ as large as possible, stopping when no such y exists.
- For a matroid, we saw that independence system $(E, \mathcal{I})$ is a matroid iff for each weight function $w \in \mathbb{R}_+^E$, the greedy algorithm leads to a set $I \in \mathcal{I}$ of maximum weight $w(I)$.
- Stated succinctly, considering $\max \{w(I) : I \in \mathcal{I}\}$, then $(E, \mathcal{I})$ is a matroid iff greedy works for this maximization.
- Can we also characterize a polymatroid in this way?
- That is, if we consider $\max \left\{ wx : x \in P_f^+ \right\}$, where P_f^+ represents the "independent vectors", is it the case that P_f^+ is a polymatroid iff greedy works for this maximization?
- Can we, ultimately, even relax things so that $w \in \mathbb{R}^E$?

Polymatroidal polyhedron and greedy

- What is the greedy solution in this setting, when $w \in \mathbb{R}^E$?
- Sort elements of E w.r.t. w so that, w.l.o.g. $E = (e_1, e_2, \ldots, e_m)$ with $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m)$.
- Let $k+1$ be the first point (if any) at which we are non-positive, i.e., $w(e_k) > 0$ and $0 \geq w(e_{k+1)}$.
- Next define partial accumulated sets E_i, for $i = 0 \ldots m$, we have w.r.t. the above sorted order:

$$E_i \stackrel{\text{def}}{=} \{e_1, e_2, \ldots e_i\} \tag{12.22}$$

(note $E_0 = \emptyset$, $f(E_0) = 0$, and E and E_i is always sorted w.r.t w).

- The greedy solution is the vector $x \in \mathbb{R}_+^E$ with elements defined as:

$$x(e_1) \stackrel{\text{def}}{=} f(E_1) = f(e_1) = f(e_1|E_0) = f(e_1|\emptyset) \tag{12.23}$$

$$x(e_i) \stackrel{\text{def}}{=} f(E_i) - f(E_{i-1}) = f(e_i|E_{i-1}) \text{ for } i = 2 \ldots k \tag{12.24}$$

$$x(e_i) \stackrel{\text{def}}{=} 0 \text{ for } i = k+1 \ldots m = |E| \tag{12.25}$$

Polymatroidal polyhedron and greedy

Theorem 12.2.2

The vector $x \in \mathbb{R}_+^E$ as previously defined using the greedy algorithm maximizes wx over P_f^+, with $w \in \mathbb{R}_+^E$, if f is submodular.

Proof.

- Consider the LP strong duality equation:

$$\max(wx : x \in P_f^+) = \min\Big(\sum_{A \subseteq E} y_A f(A) : y \in \mathbb{R}_+^{2^E}, \sum_{A \subseteq E} y_A \mathbf{1}_A \geq w\Big) \tag{12.21}$$

- Sort E by w descending, and define the following vector $y \in \mathbb{R}_+^{2^E}$ as

$$y_{E_i} \leftarrow w(e_i) - w(e_{i+1}) \text{ for } i = 1 \ldots (m-1), \tag{12.22}$$

$$y_E \leftarrow w(e_m), \text{ and} \tag{12.23}$$

$$y_A \leftarrow 0 \text{ otherwise} \tag{12.24}$$

Polymatroidal polyhedron and greedy

- Thus, restating the above results into a single complete theorem, we have a result very similar to what we saw for matroids (i.e., Theorem 9.4.1)

Theorem 12.2.2

If $f : 2^E \to \mathbb{R}_+$ is given, and P is a polytope in $\mathbb{R}_+^E$ of the form $P = \{x \in \mathbb{R}_+^E : x(A) \leq f(A), \forall A \subseteq E\}$, then the greedy solution to the problem $\max(w^\intercal x : x \in P)$ is $\forall w$ optimum iff f is monotone non-decreasing submodular (i.e., iff P is a polymatroid).

Multiple Polytopes associated with arbitrary f

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  Note that due to constraint $x(\emptyset) \leq f(\emptyset)$, we must have $f(\emptyset) \geq 0$ since if not (i.e., if $f(\emptyset) < 0$), then P_f^+ doesn't exist.
  Another form of normalization can do is:

$$f'(A) = \begin{cases} f(A) & \text{if } A \neq \emptyset \\ 0 & \text{if } A = \emptyset \end{cases} \tag{12.1}$$

  This preserves submodularity due to $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$, and if $A \cap B = \emptyset$ then r.h.s. only gets smaller when $f(\emptyset) \geq 0$.

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- We can define several polytopes:

$$P_f = \left\{x \in \mathbb{R}^E : x(S) \leq f(S), \forall S \subseteq E\right\} \tag{12.1}$$

$$P_f^+ = P_f \cap \left\{x \in \mathbb{R}^E : x \geq 0\right\} \tag{12.2}$$

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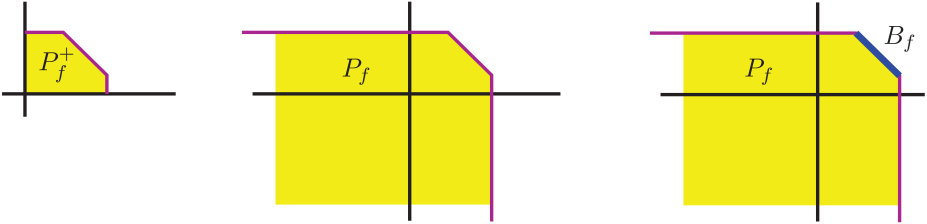
- P_f is what is sometimes called the extended polytope (sometimes notated as EP_f.
- P_f^+ is P_f restricted to the positive orthant.
- B_f is called the base polytope, analogous to the base in matroid.

Multiple Polytopes in 2D associated with f



$$P_f^+ = P_f \cap \{x \in \mathbb{R}^E : x \geq 0\} \tag{12.4}$$

$$P_f = \{x \in \mathbb{R}^E : x(S) \leq f(S), \forall S \subseteq E\} \tag{12.5}$$

$$B_f = P_f \cap \{x \in \mathbb{R}^E : x(E) = f(E)\} \tag{12.6}$$

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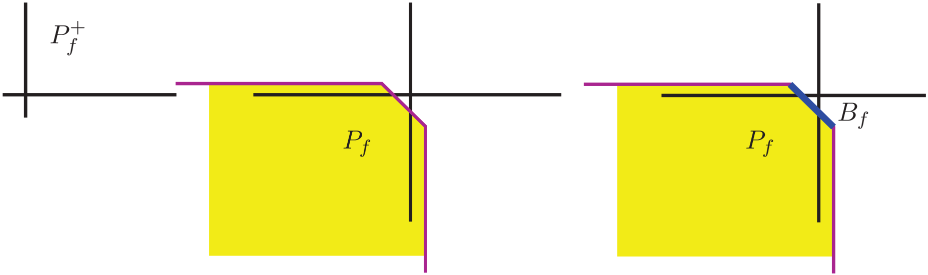

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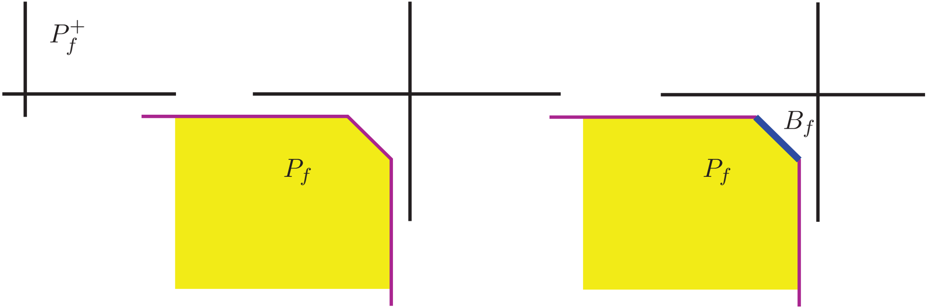

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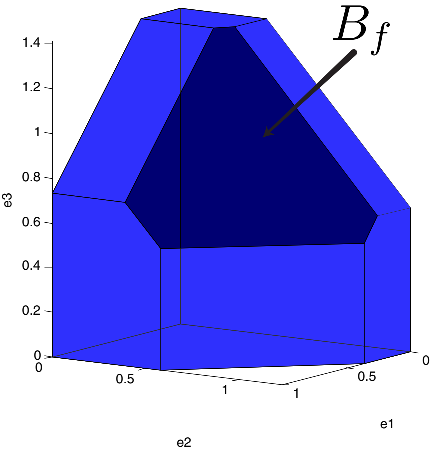
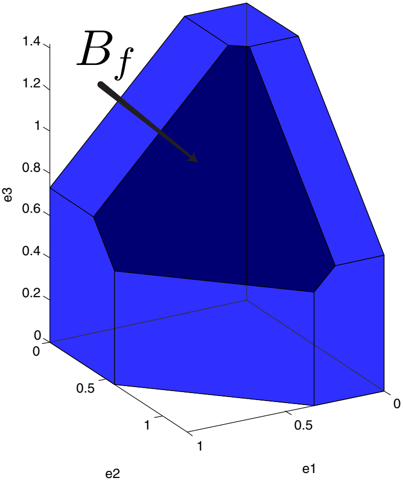
Base Polytope in 3D





$$P_f = \left\{x \in \mathbb{R}^E : x(S) \le f(S), \forall S \subseteq E\right\} \tag{12.7}$$

$$B_f = P_f \cap \left\{x \in \mathbb{R}^E : x(E) = f(E)\right\} \tag{12.8}$$

A polymatroid function's polyhedron is a polymatroid.

Theorem 12.3.1

Let f be a submodular function defined on subsets of E. For any $x \in \mathbb{R}^E$, we have:

$$\textit{rank}(x) = \max\left(y(E) : y \leq x, y \in P_f\right) = \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \quad (12.9)$$

Essentially the same theorem as Theorem 10.4.1, but note P_f rather than P_f^+. Taking $x = 0$ we get:

Corollary 12.3.2

Let f be a submodular function defined on subsets of E. We have:

$$\textit{rank}(0) = \max\left(y(E) : y \leq 0, y \in P_f\right) = \min\left(f(A) : A \subseteq E\right) \quad (12.10)$$

Proof of Theorem 12.3.1

Proof Thm 12.3.1: $\max\left(y(E) : y \leq x, y \in P_f\right) = \min\left(x(A) + f(E \setminus A) : A \subseteq E\right)$.

- Let y^* be optimal solution of the l.h.s. and let $A \subseteq E$ be any subset.

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- Then $y^*(E) = y^*(A) + y^*(E \setminus A) \leq f(A) + x(E \setminus A)$ since if $y^* \in P_f$, $y^*(A) \leq f(A)$ and since $y^* \leq x$, $y^*(E \setminus A) \leq x(E \setminus A)$. This is a form of weak duality.

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- For any $e \in E$, if $y^*(e) < x(e)$, must be some reason other than the constraint $y^* \leq x$, namely it must be that $\exists T \in \mathcal{D}(y^*)$ with $e \in T$ (i.e., e is a member of at least one of the tight sets).

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- Hence, for all $e \notin \text{sat}(y^*)$ we have $y^*(e) = x(e)$, and moreover $y^*(\text{sat}(y^*)) = f(\text{sat}(y^*))$ by definition.

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- Hence, for all $e \notin \text{sat}(y^*)$ we have $y^*(e) = x(e)$, and moreover $y^*(\text{sat}(y^*)) = f(\text{sat}(y^*))$ by definition.
- Thus $y^*(\text{sat}(y^*)) + y^*(E \setminus \text{sat}(y^*)) = f(\text{sat}(y^*)) + x(E \setminus \text{sat}(y^*))$, strong duality, showing that the two sides are equal for y^*.

∎

Greedy and P_f

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- If $\exists e$ such that $w(e) < 0$ then $\max(wx : x \in P_f) = \infty$ since we can let $x_e \to \infty$, unless we ignore the negative elements or assume $w \geq 0$.

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- In fact, we will see, in the next section, that the greedy x is a vertex of B_f.

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- Above implies that Theorem 11.4.1 can be generalized to over P_f and that greedy solution gives a point in B_f, even for arbitrary finite w.

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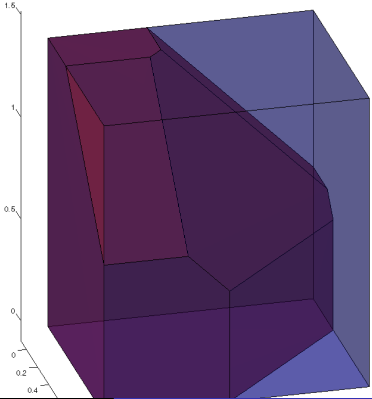

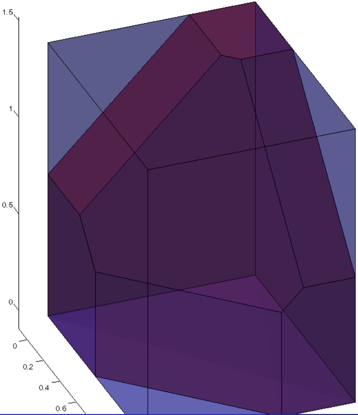

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- We formalize this next:

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- Given any arbitrary order of $E = (e_1, e_2, \ldots, e_m)$, define $E_i = (e_1, e_2, \ldots, e_i)$.

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- An extreme point of P_f is a point that is not a convex combination of two other distinct points in P_f. Equivalently, an extreme point corresponds to setting certain inequalities in the specification of P_f to be equalities, so that there is a unique single point solution.

Polymatroid extreme points

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For a given ordering $E = (e_1, \dots, e_m)$ *of* E *and a given* $E_i = (e_1, \dots, e_i)$ *and* x *generated by* E_i *using the greedy procedure* ($x(e_i) = f(e_i | E_{i-1})$)*, then* x *is an extreme point of* P_f *when* f *is submodular.*

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Proof.

- We already saw that $x \in P_f$ (Theorem 11.4.1).
- To show that x is an extreme point of P_f, note that it is the unique solution of the following system of equations

$$x(E_j) = f(E_j) \text{ for } 1 \leq j \leq i \leq m \tag{12.14}$$

$$x(e) = 0 \text{ for } e \in E \setminus E_i \tag{12.15}$$

There are $i \leq m$ equations and $i \leq m$ unknowns, and simple Gaussian elimination gives us back the x constructed via the Greedy algorithm!!

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- Also, since $x \in P_f$, for each i, we see that,

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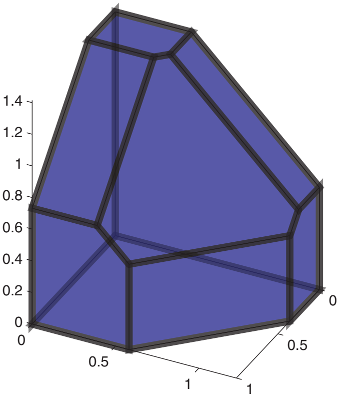
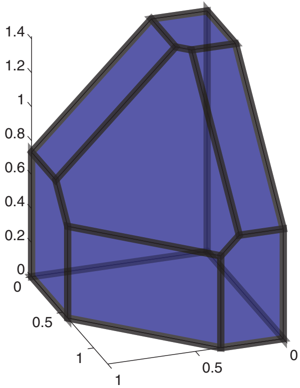
- Thus, the greedy procedure provides a modular function lower bound on f that is tight on all points E_i in the order. This can be useful in its own right, as it provides subgradients and subdifferential structure.

Polymatroid extreme points

some examples





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Polymatroid with labeled edge lengths

- Recall $f(e|A) = f(A+e) - f(A)$
- Notice how submodularity, $f(e|B) \leq f(e|A)$ for $A \subseteq B$, defines the shape of the polytope.
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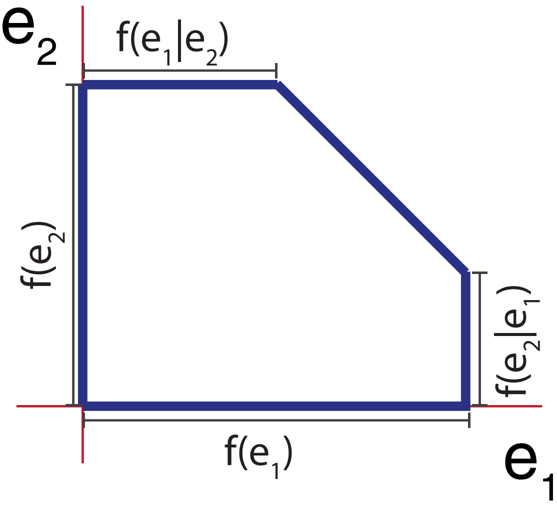

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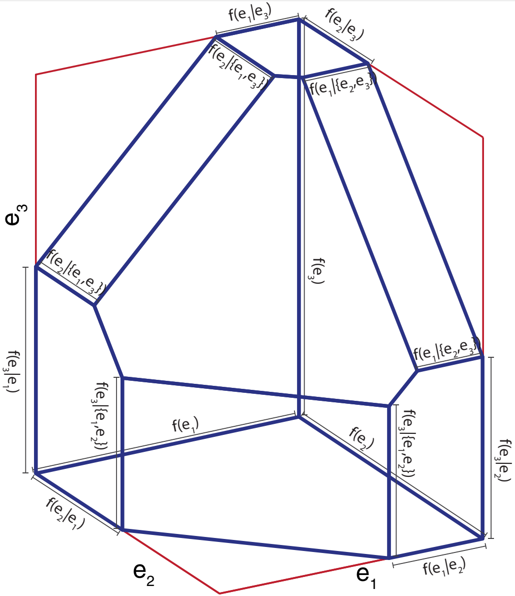

Intuition: why greedy works with polymatroids

- Given w, the goal is to find $x = (x(e_1), x(e_2))$ that maximizes $x^\intercal w = x(e_1)w(e_1) + x(e_2)w(e_2)$.
- If $w(e_2) > w(e_1)$ the upper extreme point indicated maximizes $x^\intercal w$ over $x \in P_f^+$.
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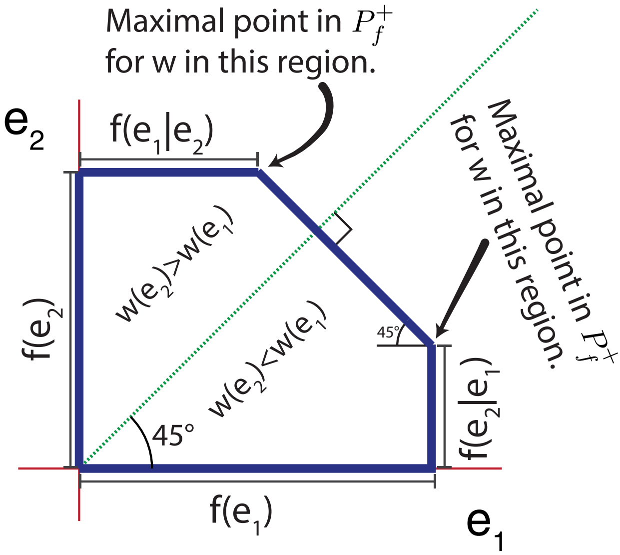

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- Thus, when we do monotone submodular maximization we find the maximum under some constraint.
- There is also a sort of dual problem that is often considered together with max, and those are minimum cover problems (to be defined).

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- We might wish to use a more general modular function $m(X)$ rather than cardinality $|X|$.

The Set Cover Problem

- Let E be a ground set and let $E_1, E_2, \ldots, E_m$ be a set of subsets.
- Let $V = \{1, 2, \ldots, m\}$ be the set of integers.
- Define $f : 2^V \to \mathbb{Z}_+$ as $f(X) = |\bigcup_{v \in X} E_v|$
- Then f is the set cover function. As we say, f is monotone submodular (a polymatroid).
- The set cover problem asks for the smallest subset X of V such that $f(X) = |E|$ (smallest subset of the subsets of E) where E is still covered. I.e.,

$$\text{minimize} |X| \text{ subject to } f(X) \geq |E| \tag{12.18}$$

- We might wish to use a more general modular function $m(X)$ rather than cardinality $|X|$.
- This problem is NP-hard, and Feige in 1998 showed that it cannot be approximated with a ratio better than $(1 - \epsilon) \log n$ unless NP is slightly superpolynomial ($n^{O(\log \log n)}$).

What About Non-monotone

- So even simple case of cardinality constrained submodular function maximization is NP-hard.
- This will be true of most submodular max (and related) problems.
- Hence, the only hope is approximation algorithms. Question is, what is the tradeoff between running time and approximation quality, and is it possible to get tight bounds (i.e., an algorithm that achieves an approximation ratio, and a proof that one can't do better than that unless some extremely unlike event were to be true, such as P=NP).

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- The max k cover problem asks, given a k, what sized k set of sets X can we choose that covers the most? I.e., that maximizes $f(X)$ as in:

$$\max f(X) \text{ subject to } |X| \leq k \tag{12.19}$$

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- An important result by Nemhauser et. al. (1978) states that for normalized ($f(\emptyset) = 0$) monotone submodular functions (i.e., polymatroids) can be approximately maximized using a simple greedy algorithm.

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- An important result by Nemhauser et. al. (1978) states that for normalized ($f(\emptyset) = 0$) monotone submodular functions (i.e., polymatroids) can be approximately maximized using a simple greedy algorithm.
- Starting with $S_0 = \emptyset$, we repeat the following greedy step for $i = 0 \ldots (k-1)$:

$$S_{i+1} = S_i \cup \left\{ \operatorname*{argmax}_{v \in V \setminus S_i} f(S_i \cup \{v\}) \right\} \tag{12.20}$$

The Greedy Algorithm for Submodular Max

A bit more precisely:

Algorithm 1: The Greedy Algorithm

```
Set S_0 ← ∅ ;
for i ← 0 ... |E| − 1 do
    Choose v_i as follows:
      v_i ∈ argmax_{v∈V\S_i} f({v}|S_i) = argmax_{v∈V\S_i} f(S_i ∪ {v}) ;
    Set S_{i+1} ← S_i ∪ {v_i} ;
```

Set $S_0 \leftarrow \emptyset$;

for $i \leftarrow 0 \ldots |E| - 1$ **do**

Choose v_i as follows:

$$v_i \in \operatorname{argmax}_{v \in V \setminus S_i} f(\{v\}|S_i) = \operatorname{argmax}_{v \in V \setminus S_i} f(S_i \cup \{v\}) \;;$$

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Given a polymatroid function f , the above greedy algorithm returns sets S_i such that for each i we have $f(S_i) \geq (1 - 1/e) \max_{|S| \leq i} f(S)$.

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- To approximately find $A^* \in \operatorname{argmax} \{f(A) : |A| \leq k\}$, we repeat the greedy step until $k = i + 1$:
- Again, since this generalizes max k-cover, Feige (1998) showed that this can't be improved. Unless $P = NP$, no polynomial time algorithm can do better than $(1 - 1/e + \epsilon)$ for any $\epsilon > 0$.

The Greedy Algorithm: $1 - 1/e$ intuition.

- At step $i < k$, greedy chooses v_i to maximize $f(v|S_i)$.

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$$\exists v \in V \setminus S_i : f(v|S_i) = f(S_i + v|S_i) \geq \frac{1}{k}(\text{OPT} - f(S_i)) \tag{12.21}$$

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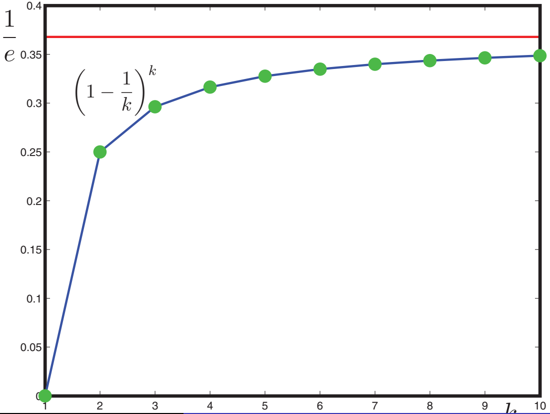

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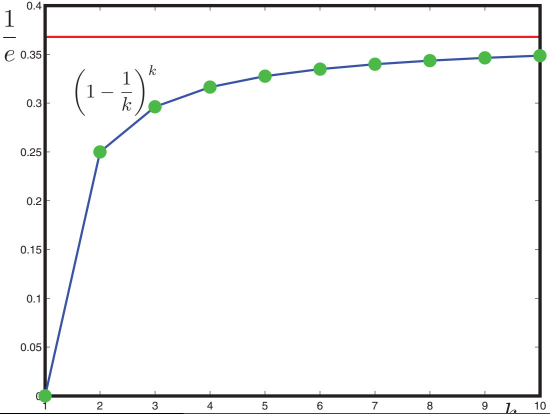

Equation (12.30) will show that Equation (12.21) $\Rightarrow$:

$$\begin{aligned} &\text{OPT} - f(S_{i+1}) \\ &\qquad \leq (1 - 1/k)(\text{OPT} - f(S_i)) \\ &\Rightarrow \text{OPT} - f(S_k) \\ &\qquad \leq (1 - 1/k)^k \text{OPT} \\ &\qquad \leq 1/e\text{OPT} \\ &\Rightarrow \text{OPT}(1 - 1/e) \leq f(S_k) \end{aligned}$$

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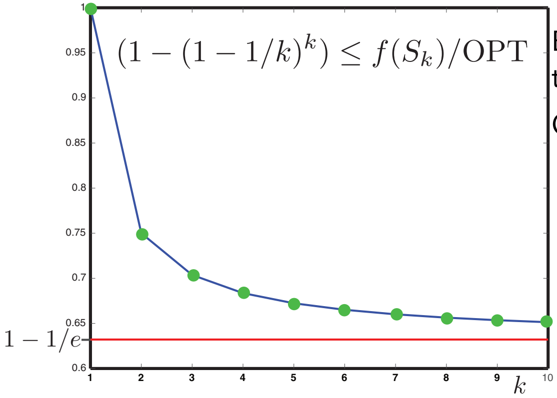

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Cardinality Constrained Polymatroid Max Theorem

Theorem 12.5.2 (Nemhauser et al. 1978)

Given non-negative monotone submodular function $f : 2^V \to \mathbb{R}_+$, *define* $\{S_i\}_{i \geq 0}$ *to be the chain formed by the greedy algorithm (Eqn. (12.20)). Then for all* $k, \ell \in \mathbb{Z}_{++}$, *we have:*

$$f(S_\ell) \geq (1 - e^{-\ell/k}) \max_{S:|S| \leq k} f(S) \tag{12.22}$$

and in particular, for $\ell = k$, *we have* $f(S_k) \geq (1 - 1/e) \max_{S:|S| \leq k} f(S)$.

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- Intuitively, bound should get worse when $\ell < k$ and get better when $\ell > k$.

Cardinality Constrained Polymatroid Max Theorem

Proof of Theorem 12.5.2.

...

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- Then the following inequalities (on the next slide) follow:

...

Cardinality Constrained Polymatroid Max Theorem

. . . proof of Theorem 12.5.2 cont.

. . .

Cardinality Constrained Polymatroid Max Theorem

. . . proof of Theorem 12.5.2 cont.

- For all $i < \ell$, we have

$$f(S^*)$$

. . .

Cardinality Constrained Polymatroid Max Theorem

. . . proof of Theorem 12.5.2 cont.

- For all $i < \ell$, we have

$$f(S^*) \leq f(S^* \cup S_i)$$

. . .

Cardinality Constrained Polymatroid Max Theorem

. . . proof of Theorem 12.5.2 cont.

- For all $i < \ell$, we have

$$f(S^*) \leq f(S^* \cup S_i) = f(S_i) + f(S^*|S_i) \tag{12.23}$$

. . .

Cardinality Constrained Polymatroid Max Theorem

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$$\leq f(S_i) + \sum_{v \in S^*} f(v|S_i) \tag{12.25}$$

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. . .

Cardinality Constrained Polymatroid Max Theorem

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$$= f(S_i) + kf(S_{i+1}|S_i) \tag{12.27}$$

- Therefore, we have Equation 12.21, i.e.,:

$$f(S^*) - f(S_i) \leq kf(S_{i+1}|S_i) = k(f(S_{i+1}) - f(S_i)) \tag{12.28}$$

...

Cardinality Constrained Polymatroid Max Theorem

. . . proof of Theorem 12.5.2 cont.

Cardinality Constrained Polymatroid Max Theorem

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- Define gap $\delta_i \triangleq f(S^*) - f(S_i)$, so $\delta_i - \delta_{i+1} = f(S_{i+1}) - f(S_i)$,

Cardinality Constrained Polymatroid Max Theorem

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- The relationship between δ_0 and δ_ℓ is then

$$\delta_l \leq (1 - \frac{1}{k})^\ell \delta_0 \tag{12.31}$$

Cardinality Constrained Polymatroid Max Theorem

. . . proof of Theorem 12.5.2 cont.

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$$\delta_i \leq k(\delta_i - \delta_{i+1}) \tag{12.29}$$

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$$\delta_{i+1} \leq (1 - \frac{1}{k})\delta_i \tag{12.30}$$

- The relationship between δ_0 and δ_ℓ is then

$$\delta_l \leq (1 - \frac{1}{k})^\ell \delta_0 \tag{12.31}$$

- Now, $\delta_0 = f(S^*) - f(\emptyset) \leq f(S^*)$ since $f \geq 0$.

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- Also, by variational bound $1 - x \leq e^{-x}$ for $x \in \mathbb{R}$, we have

$$\delta_\ell \leq (1 - \frac{1}{k})^\ell \delta_0 \leq e^{-\ell/k} f(S^*) \tag{12.32}$$

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- So solution, in the worst case, quickly gets very good. Typical/practical case is much better.

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- We describe it next:

Minoux's Accelerated Greedy for Submodular Functions

- At stage i in the algorithm, we have a set of gains $f(v|S_i)$ for all $v \notin S_i$. Store these values $\alpha_v \leftarrow f(v|S_i)$ in sorted priority queue.

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- For $v \notin S_{i+1}$ we have $f(v|S_{i+1}) \leq f(v|S_i)$ by submodularity.
- Therefore, if we find a v' such that $f(v'|S_{i+1}) \geq \alpha_v$ for all $v \neq v'$, then since

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- Strategy is: find the $\text{argmax}_{v' \in V \setminus S_{i+1}} \alpha_{v'}$, and then compute the real $f(v'|S_{i+1})$. If it is greater than all other α_v's then that's the next greedy step. Otherwise, replace $\alpha_{v'}$ with its real value, resort $(O(\log n))$, and repeat.

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- Algorithm has been rediscovered (I think) independently (CELF - cost-effective lazy forward selection, Leskovec et al., 2007)
- Can be used used for "big data" sets (e.g., social networks, selecting blogs of greatest influence, document summarization, etc.).

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- If a popped item is fresh, it must be the maximum — this can happen if, at given iteration, v was first popped and neither fresh nor maximum so placed back in the queue, and it then percolates back to the top at which point it is fresh — thereby avoid extra queue check.

Minoux's Accelerated Greedy Algorithm Submodular Max

Algorithm 2: Minoux's Accelerated Greedy Algorithm

1 Set $S_0 \leftarrow \emptyset$; $i \leftarrow 0$; Initialize priority queue Q ;
2 **for** $v \in E$ **do**
3   INSERT$(Q, f(v))$
4 **repeat**
5     $(v, \alpha) \leftarrow$ pop(Q) ;
6   **if** α *not "fresh"* **then**
7   recompute $\alpha \leftarrow f(v|S_i)$
8   **if** *(popped α in line 5 was "fresh") OR ($\alpha \geq$ max(Q))* **then**
9   Set $S_{i+1} \leftarrow S_i \cup \{v\}$;
10         $i \leftarrow i + 1$;
11   else
12   insert$(Q, (v, \alpha))$
13 **until** $i = |E|$;

(Minimum) Submodular Set Cover

- Given polymatroid f, goal is to find a covering set of minimum cost:

$$S^* \in \operatorname*{argmin}_{S \subseteq V} |S| \text{ such that } f(S) \geq \alpha \tag{12.38}$$

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- Note that this immediately generalizes standard set cover, in which case $f(A)$ is the cardinality of the union of sets indexed by A.
- Greedy Algorithm: Pick the first chain item S_i chosen by aforementioned greedy algorithm such that $f(S_i) \geq \alpha$ and output that as solution.

(Minimum) Submodular Set Cover: Approximation Analysis

- For integer valued f, this greedy algorithm an $O(\log(\max_{s \in V} f(\{s\})))$ approximation. Let S^* be optimal, and S^{G} be greedy solution, then

$$|S^{\mathsf{G}}| \leq |S^*| H(\max_{s \in V} f(\{s\})) = |S^*| O(\log_e(\max_{s \in V} f(\{s\}))) \qquad (12.40)$$

where H is the harmonic function, i.e., $H(d) = \sum_{i=1}^{d} (1/i)$.

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- If f is not integral value, then bounds we get are of the form:

$$|S^{\mathsf{G}}| \leq |S^*| \left(1 + \log_e \frac{f(V)}{f(V) - f(S_{T-1})}\right) \tag{12.41}$$

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- Set cover is hard to approximate with a factor better than $(1 - \epsilon) \log \alpha$, where α is the desired cover constraint.

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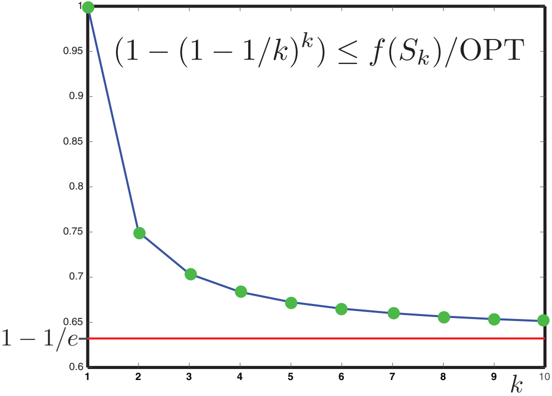
- Submodular cover: min. $|S|$ s.t. $f(S) \geq \alpha$.
- Minoux's accelerated greedy trick.

The Greedy Algorithm: $1 - 1/e$ intuition.

- At step $i < k$, greedy chooses v_i to maximize $f(v|S_i)$.
- Let S^* be optimal solution (of size k) and OPT $= f(S^*)$. By submodularity, we will show:

$$\exists v \in V \setminus S_i : f(v|S_i) = f(S_i + v|S_i) \geq \frac{1}{k}(\text{OPT} - f(S_i)) \qquad (12.21)$$



Equation (12.30) will show that Equation (12.21) $\Rightarrow$:

$$\begin{aligned}
&\text{OPT} - f(S_{i+1}) \\
&\qquad \leq (1-1/k)(\text{OPT} - f(S_i)) \\
\Rightarrow \;& \text{OPT} - f(S_k) \\
&\qquad \leq (1-1/k)^k \text{OPT} \\
&\qquad \leq 1/e\text{OPT} \\
\Rightarrow \;& \text{OPT}(1-1/e) \leq f(S_k)
\end{aligned}$$

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- See problem 5, homework 4.