Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture 13 —

http://www.ee.washington.edu/people/faculty/bilmes/classes/ee563_spring_2018/

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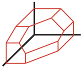
May 9th, 2018

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

$= f(A_r) + 2f(C) + f(B_r)$ $\quad = f(A_r) + f(C) + f(B_r)$ $\quad = f(A \cap B)$



Cumulative Outstanding Reading

- Read chapter 1 from Fujishige's book.
- Read chapter 2 from Fujishige's book.
- Read chapter 3 from Fujishige's book.
- Read chapter 4 from Fujishige's book.

Announcements, Assignments, and Reminders

- Next homework is posted on canvas. Due Thursday 5/10, 11:59pm.
- As always, if you have any questions about anything, please ask then via our discussion board (`https://canvas.uw.edu/courses/1216339/discussion_topics`). Can meet at odd hours via zoom (send message on canvas to schedule time to chat).

Class Road Map - EE563

- L1(3/26): Motivation, Applications, & Basic Definitions,
- L2(3/28): Machine Learning Apps (diversity, complexity, parameter, learning target, surrogate).
- L3(4/2): Info theory exs, more apps, definitions, graph/combinatorial examples
- L4(4/4): Graph and Combinatorial Examples, Matrix Rank, Examples and Properties, visualizations
- L5(4/9): More Examples/Properties/ Other Submodular Defs., Independence,
- L6(4/11): Matroids, Matroid Examples, Matroid Rank, Partition/Laminar Matroids
- L7(4/16): Laminar Matroids, System of Distinct Reps, Transversals, Transversal Matroid, Matroid Representation, Dual Matroids
- L8(4/18): Dual Matroids, Other Matroid Properties, Combinatorial Geometries, Matroids and Greedy.
- L9(4/23): Polyhedra, Matroid Polytopes, Matroids $\rightarrow$ Polymatroids
- L10(4/29): Matroids $\rightarrow$ Polymatroids, Polymatroids, Polymatroids and Greedy,
- L11(4/30): Polymatroids, Polymatroids and Greedy
- L12(5/2): Polymatroids and Greedy, Extreme Points, Cardinality Constrained Maximization
- L13(5/7): Constrained Submodular Maximization
- L14(5/9):
- L15(5/14):
- L16(5/16):
- L17(5/21):
- L18(5/23):
- L–(5/28): Memorial Day (holiday)
- L19(5/30):
- L21(6/4): Final Presentations maximization.

Last day of instruction, June 1st. Finals Week: June 2-8, 2018.

Multiple Polytopes associated with arbitrary f

- Given an arbitrary submodular function $f : 2^V \to R$ (not necessarily a polymatroid function, so it need not be positive, monotone, etc.).
- If $f(\emptyset) \neq 0$, can set $f'(A) = f(A) - f(\emptyset)$ without destroying submodularity. This does not change any minima, (i.e., $\text{argmin}_A f(A) = \text{argmin}_{A'} f'(A)$) so we often assume all functions are normalized $f(\emptyset) = 0$.
- We can define several polytopes:

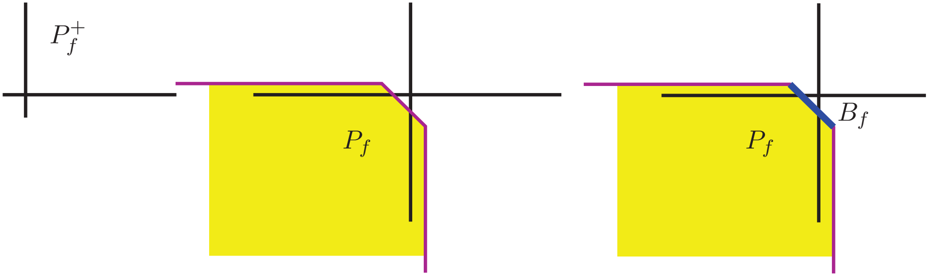
$$P_f = \left\{x \in \mathbb{R}^E : x(S) \leq f(S), \forall S \subseteq E\right\} \tag{13.1}$$

$$P_f^+ = P_f \cap \left\{x \in \mathbb{R}^E : x \geq 0\right\} \tag{13.2}$$

$$B_f = P_f \cap \left\{x \in \mathbb{R}^E : x(E) = f(E)\right\} \tag{13.3}$$

- P_f is what is sometimes called the extended polytope (sometimes notated as EP_f.
- P_f^+ is P_f restricted to the positive orthant.
- B_f is called the base polytope, analogous to the base in matroid.

Multiple Polytopes in 2D associated with f



$$P_f^+ = P_f \cap \left\{x \in \mathbb{R}^E : x \geq 0\right\} \tag{13.1}$$

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A polymatroid function's polyhedron is a polymatroid.

Theorem 13.2.1

Let f be a submodular function defined on subsets of E. For any $x \in \mathbb{R}^E$, we have:

$$\textit{rank}(x) = \max\left(y(E) : y \leq x, y \in P_f\right) = \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \tag{13.1}$$

Essentially the same theorem as Theorem **??**, but note P_f rather than P_f^+. Taking $x = 0$ we get:

Corollary 13.2.2

Let f be a submodular function defined on subsets of E. We have:

$$\textit{rank}(0) = \max\left(y(E) : y \leq 0, y \in P_f\right) = \min\left(f(A) : A \subseteq E\right) \tag{13.2}$$

Polymatroid extreme points

Theorem 13.2.1

For a given ordering $E = (e_1, \dots, e_m)$ of E and a given $E_i = (e_1, \dots, e_i)$ and x generated by E_i using the greedy procedure ($x(e_i) = f(e_i|E_{i-1})$), then x is an extreme point of P_f when f is submodular.

Proof.

- We already saw that $x \in P_f$ (Theorem ??).
- To show that x is an extreme point of P_f, note that it is the unique solution of the following system of equations

$$x(E_j) = f(E_j) \text{ for } 1 \leq j \leq i \leq m \tag{13.4}$$

$$x(e) = 0 \text{ for } e \in E \setminus E_i \tag{13.5}$$

There are $i \leq m$ equations and $i \leq m$ unknowns, and simple Gaussian elimination gives us back the x constructed via the Greedy algorithm!!

Polymatroid extreme points

- Moreover, we have (and will ultimately prove)

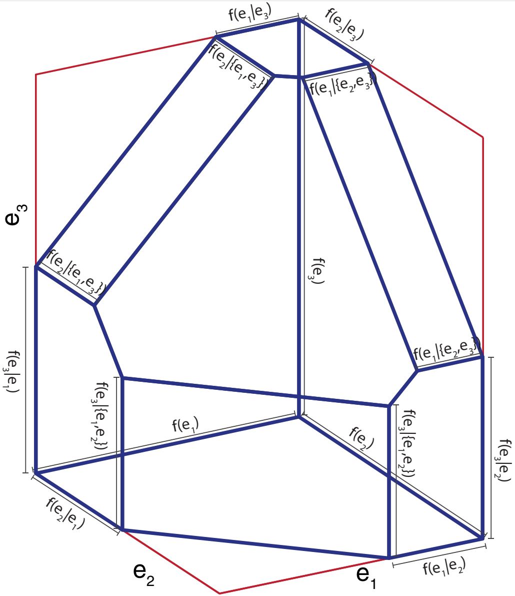
Corollary 13.2.2

If x is an extreme point of P_f and $B \subseteq E$ is given such that $\text{supp}(x) = \{e \in E : x(e) \neq 0\} \subseteq B \subseteq \cup(A : x(A) = f(A)) = \text{sat}(x)$, then x is generated using greedy by some ordering of B.

- Note, $\text{sat}(x) = \text{cl}(x) = \cup(A : x(A) = f(A))$ is also called the closure of x (recall that sets A such that $x(A) = f(A)$ are called tight, and such sets are closed under union and intersection, as seen in Lecture 10, Theorem ??)
- Thus, $\text{cl}(x)$ is a tight set.
- Also, $\text{supp}(x) = \{e \in E : x(e) \neq 0\}$ is called the support of x.
- For arbitrary x, $\text{supp}(x)$ is not necessarily tight, but for an extreme point, $\text{supp}(x)$ is.

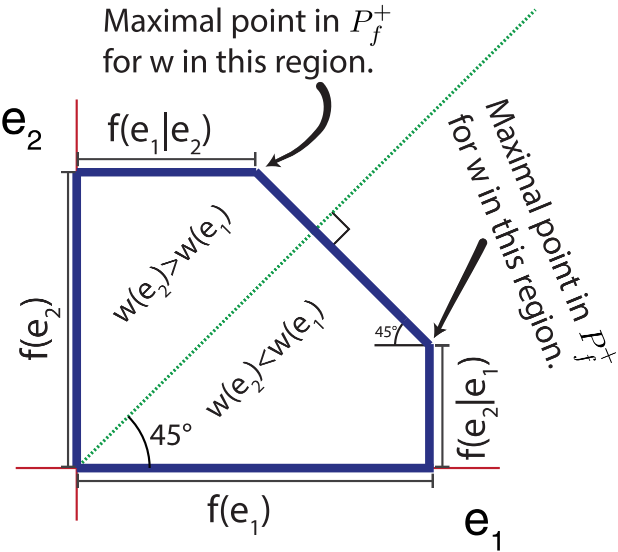
Polymatroid with labeled edge lengths

- Recall $f(e|A) = f(A+e) - f(A)$
- Notice how submodularity, $f(e|B) \leq f(e|A)$ for $A \subseteq B$, defines the shape of the polytope.
- In fact, we have strictness here $f(e|B) < f(e|A)$ for $A \subset B$.
- Also, consider how the greedy algorithm proceeds along the edges of the polytope.



Intuition: why greedy works with polymatroids

- Given w, the goal is to find $x = (x(e_1), x(e_2))$ that maximizes $x^\intercal w = x(e_1)w(e_1) + x(e_2)w(e_2)$.
- If $w(e_2) > w(e_1)$ the upper extreme point indicated maximizes $x^\intercal w$ over $x \in P_f^+$.
- If $w(e_2) < w(e_1)$ the lower extreme point indicated maximizes $x^\intercal w$ over $x \in P_f^+$.



The Greedy Algorithm for Submodular Max

A bit more precisely:

Algorithm 1: The Greedy Algorithm

Set $S_0 \leftarrow \emptyset$;
for $i \leftarrow 0 \ldots |E| - 1$ **do**
    Choose v_i as follows:
    $v_i \in \operatorname{argmax}_{v \in V \setminus S_i} f(\{v\}|S_i) = \operatorname{argmax}_{v \in V \setminus S_i} f(S_i \cup \{v\})$;
    Set $S_{i+1} \leftarrow S_i \cup \{v_i\}$;

Greedy Algorithm for Card. Constrained Submodular Max

- This algorithm has a guarantee

Theorem 13.2.1

Given a polymatroid function f , the above greedy algorithm returns sets S_i such that for each i we have $f(S_i) \geq (1 - 1/e) \max_{|S| \leq i} f(S)$.

- To approximately find $A^* \in \operatorname{argmax} \{f(A) : |A| \leq k\}$, we repeat the greedy step until $k = i + 1$:
- Again, since this generalizes max k-cover, Feige (1998) showed that this can't be improved. Unless $P = NP$, no polynomial time algorithm can do better than $(1 - 1/e + \epsilon)$ for any $\epsilon > 0$.

The Greedy Algorithm: $1 - 1/e$ intuition.

- At step $i < k$, greedy chooses v_i to maximize $f(v|S_i)$.

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- At step $i < k$, greedy chooses v_i to maximize $f(v|S_i)$.
- Let S^* be optimal solution (of size k) and $\text{OPT} = f(S^*)$. By submodularity, we will show:

$$\exists v \in V \setminus S_i : f(v|S_i) = f(S_i + v|S_i) \geq \frac{1}{k}(\text{OPT} - f(S_i)) \tag{13.1}$$

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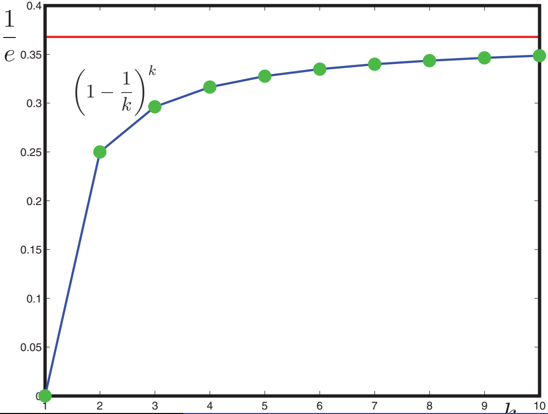

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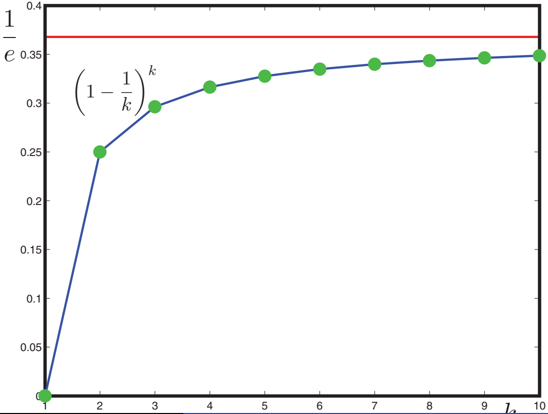

Equation (13.10) will show that Equation (13.1) $\Rightarrow$:

$$\text{OPT} - f(S_{i+1}) \leq (1 - 1/k)(\text{OPT} - f(S_i))$$

$$\Rightarrow \text{OPT} - f(S_k) \leq (1 - 1/k)^k \text{OPT} \leq 1/e\,\text{OPT}$$

$$\Rightarrow \text{OPT}(1 - 1/e) \leq f(S_k)$$

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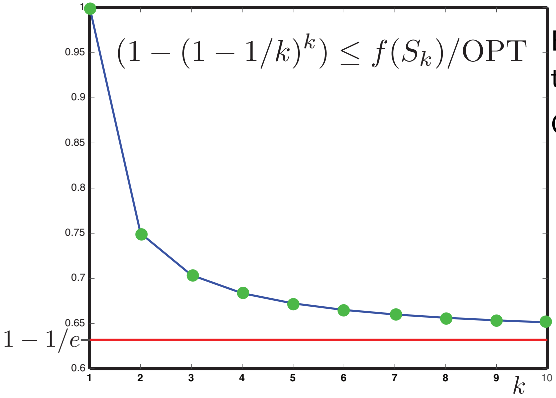

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Cardinality Constrained Polymatroid Max Theorem

Theorem 13.3.1 (Nemhauser et al. 1978)

Given non-negative monotone submodular function $f : 2^V \to \mathbb{R}_+$, define $\{S_i\}_{i \geq 0}$ to be the chain formed by the greedy algorithm (Eqn. (??)). Then for all $k, \ell \in \mathbb{Z}_{++}$, we have:

$$f(S_\ell) \geq (1 - e^{-\ell/k}) \max_{S:|S| \leq k} f(S) \tag{13.2}$$

and in particular, for $\ell = k$, we have $f(S_k) \geq (1 - 1/e) \max_{S:|S| \leq k} f(S)$.

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- k is size of optimal set, i.e., OPT $= f(S^*)$ with $|S^*| = k$
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- ℓ is size of set we are choosing (i.e., we choose S_ℓ from greedy chain).
- Bound is how well does S_ℓ (of size ℓ) do relative to S^*, the optimal set of size k.
- Intuitively, bound should get worse when $\ell < k$ and get better when $\ell > k$.

Cardinality Constrained Polymatroid Max Theorem

Proof of Theorem 13.3.1.

...

Cardinality Constrained Polymatroid Max Theorem

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- Fix ℓ (number of items greedy will chose) and k (size of optimal set to compare against).

...

Cardinality Constrained Polymatroid Max Theorem

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- w.l.o.g. assume $|S^*| = k$.
- Order $S^* = (v_1^*, v_2^*, \ldots, v_k^*)$ arbitrarily.
- Let $S_i = (v_1, v_2, \ldots, v_i)$ be the greedy order chain chosen by the algorithm, for $i \in \{1, 2, \ldots, \ell\}$.

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- Let $S_i = (v_1, v_2, \ldots, v_i)$ be the greedy order chain chosen by the algorithm, for $i \in \{1, 2, \ldots, \ell\}$.
- Then the following inequalities (on the next slide) follow:

...

Cardinality Constrained Polymatroid Max Theorem

... proof of Theorem 13.3.1 cont.

...

Cardinality Constrained Polymatroid Max Theorem

. . . proof of Theorem 13.3.1 cont.

- For all $i < \ell$, we have

$$f(S^*)$$

. . .

Cardinality Constrained Polymatroid Max Theorem

. . . proof of Theorem 13.3.1 cont.

- For all $i < \ell$, we have

$$f(S^*) \leq f(S^* \cup S_i)$$

. . .

Cardinality Constrained Polymatroid Max Theorem

. . . proof of Theorem 13.3.1 cont.

- For all $i < \ell$, we have

$$f(S^*) \leq f(S^* \cup S_i) = f(S_i) + f(S^*|S_i) \tag{13.3}$$

. . .

Cardinality Constrained Polymatroid Max Theorem

...proof of Theorem 13.3.1 cont.

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$$= f(S_i) + \sum_{j=1}^{k} f(v_j^*|S_i \cup \{v_1^*, v_2^*, \ldots, v_{j-1}^*\}) \tag{13.4}$$

...

Cardinality Constrained Polymatroid Max Theorem

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$$\leq f(S_i) + \sum_{v \in S^*} f(v_{i+1}|S_i)$$

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$$\leq f(S_i) + \sum_{v \in S^*} f(v_{i+1}|S_i) = f(S_i) + \sum_{v \in S^*} f(S_{i+1}|S_i) \tag{13.6}$$

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Cardinality Constrained Polymatroid Max Theorem

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Cardinality Constrained Polymatroid Max Theorem

... proof of Theorem 13.3.1 cont.

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$$\leq f(S_i) + \sum_{v \in S^*} f(v_{i+1}|S_i) = f(S_i) + \sum_{v \in S^*} f(S_{i+1}|S_i) \tag{13.6}$$

$$= f(S_i) + k f(S_{i+1}|S_i) \tag{13.7}$$

- Therefore, we have Equation 13.1, i.e.,:

$$f(S^*) - f(S_i) \leq k f(S_{i+1}|S_i) = k(f(S_{i+1}) - f(S_i)) \tag{13.8}$$

...

Cardinality Constrained Polymatroid Max Theorem

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- Now, $\delta_0 = f(S^*) - f(\emptyset) \leq f(S^*)$ since $f \geq 0$.
- Also, by variational bound $1 - x \leq e^{-x}$ for $x \in \mathbb{R}$, we have
$$\delta_\ell \leq (1 - \frac{1}{k})^\ell \delta_0 \leq e^{-\ell/k} f(S^*) \tag{13.12}$$

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- With $\ell = k$, when picking k items, greedy gets $(1 - 1/e) \approx 0.6321$ bound. This means that if S_k is greedy solution of size k, and S^* is an optimal solution of size k, $f(S_k) \geq (1 - 1/e) f(S^*) \approx 0.6321 f(S^*)$.

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- So solution, in the worst case, quickly gets very good. Typical/practical case is much better.

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- We describe it next:

Minoux's Accelerated Greedy for Submodular Functions

- At stage i in the algorithm, we have a set of gains $f(v|S_i)$ for all $v \notin S_i$. Store these values $\alpha_v \leftarrow f(v|S_i)$ in sorted priority queue.

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- For $v \notin S_{i+1}$ we have $f(v|S_{i+1}) \leq f(v|S_i)$ by submodularity.
- Therefore, if we find a v' such that $f(v'|S_{i+1}) \geq \alpha_v$ for all $v \neq v'$, then since

$$f(v'|S_{i+1}) \geq \alpha_v = f(v|S_i) \geq f(v|S_{i+1}) \tag{13.14}$$

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- Strategy is: find the $\operatorname{argmax}_{v' \in V \setminus S_{i+1}} \alpha_{v'}$, and then compute the real $f(v'|S_{i+1})$. If it is greater than all other α_v's then that's the next greedy step. Otherwise, replace $\alpha_{v'}$ with its real value, resort $(O(\log n))$, and repeat.

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- Algorithm has been rediscovered (I think) independently (CELF - cost-effective lazy forward selection, Leskovec et al., 2007)
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- If a popped item is fresh, it must be the maximum — this can happen if, at given iteration, v was first popped and neither fresh nor maximum so placed back in the queue, and it then percolates back to the top at which point it is fresh — thereby avoid extra queue check.

Minoux's Accelerated Greedy Algorithm Submodular Max

Algorithm 2: Minoux's Accelerated Greedy Algorithm

```
1  Set S_0 ← ∅ ; i ← 0 ; Initialize priority queue Q ;
2  for v ∈ E do
3      INSERT(Q, f(v))
4  repeat
5      (v, α) ← pop(Q) ;
6      if α not "fresh" then
7          recompute α ← f(v|S_i)
8      if (popped α in line 5 was "fresh") OR (α ≥ max(Q)) then
9          Set S_{i+1} ← S_i ∪ {v} ;
10         i ← i + 1 ;
11     else
12         insert(Q, (v, α))
13 until i = |E|;
```

(Minimum) Submodular Set Cover

- Given polymatroid f, goal is to find a covering set of minimum cost:

$$S^* \in \operatorname*{argmin}_{S \subseteq V} |S| \text{ such that } f(S) \geq \alpha \tag{13.18}$$

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- Normally take $\alpha = f(V)$ but defining $f'(A) = \min\{f(A), \alpha\}$ we can take any α. Hence, we have equivalent formulation:

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- Note that this immediately generalizes standard set cover, in which case $f(A)$ is the cardinality of the union of sets indexed by A.
- Greedy Algorithm: Pick the first chain item S_i chosen by aforementioned greedy algorithm such that $f(S_i) \geq \alpha$ and output that as solution.

(Minimum) Submodular Set Cover: Approximation Analysis

- For integer valued f, this greedy algorithm an $O(\log(\max_{s \in V} f(\{s\})))$ approximation. Let S^* be optimal, and S^{G} be greedy solution, then

$$|S^{\mathsf{G}}| \leq |S^*| H(\max_{s \in V} f(\{s\})) = |S^*| O(\log_e(\max_{s \in V} f(\{s\}))) \tag{13.20}$$

where H is the harmonic function, i.e., $H(d) = \sum_{i=1}^{d} (1/i)$.

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- If f is not integral value, then bounds we get are of the form:

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- Set cover is hard to approximate with a factor better than $(1-\epsilon)\log \alpha$, where α is the desired cover constraint.

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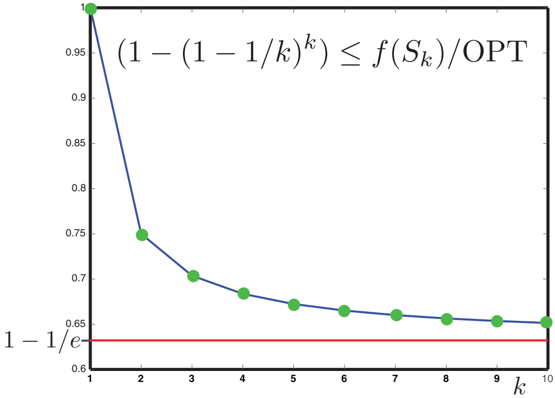
- Submodular cover: min. $|S|$ s.t. $f(S) \geq \alpha$.
- Minoux's accelerated greedy trick.

The Greedy Algorithm: $1 - 1/e$ intuition.

- At step $i < k$, greedy chooses v_i to maximize $f(v|S_i)$.
- Let S^* be optimal solution (of size k) and OPT $= f(S^*)$. By submodularity, we will show:

$$\exists v \in V \setminus S_i : f(v|S_i) = f(S_i + v|S_i) \geq \frac{1}{k}(\text{OPT} - f(S_i)) \tag{13.1}$$



Equation (13.10) will show that Equation (13.1) $\Rightarrow$:

$$\begin{aligned}
&\text{OPT} - f(S_{i+1}) \\
&\qquad \leq (1 - 1/k)(\text{OPT} - f(S_i)) \\
&\Rightarrow \text{OPT} - f(S_k) \\
&\qquad \leq (1 - 1/k)^k \text{OPT} \\
&\qquad \leq 1/e\text{OPT} \\
&\Rightarrow \text{OPT}(1 - 1/e) \leq f(S_k)
\end{aligned}$$

Randomized greedy

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- Suppose the following holds:

$$E[f(a_{i+1}|A_i)] \geq \frac{f(OPT) - f(A_i)}{k} \tag{13.22}$$

where $A_i = (a_1, a_2, \ldots, a_i)$ are the first i elements chosen by the strategy.

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- It will be remembered the notion of "partial dependence" within polymatroid functions.

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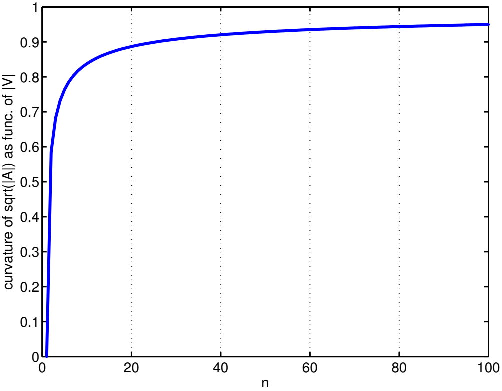
Curvature of $f(S) = \sqrt{|S|}$ as function of $|V| = n$



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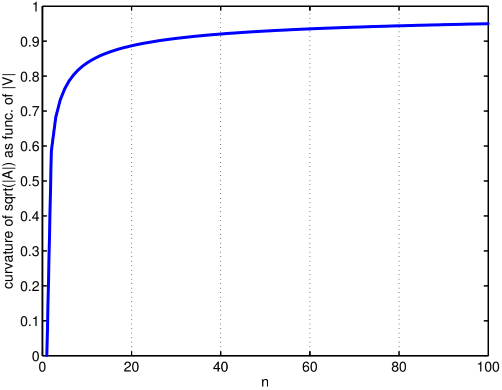

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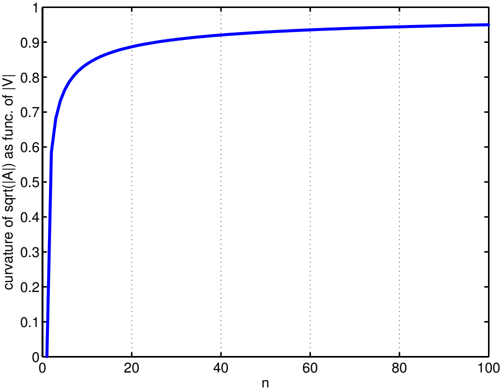

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- Functions of the form $f(S) = \sqrt{m(S)}$ where $m : V \to \mathbb{R}_+$, approximation worse with n if $\min_{i,j} |m(i) - m(j)|$ has a fixed lower bound with increasing n.

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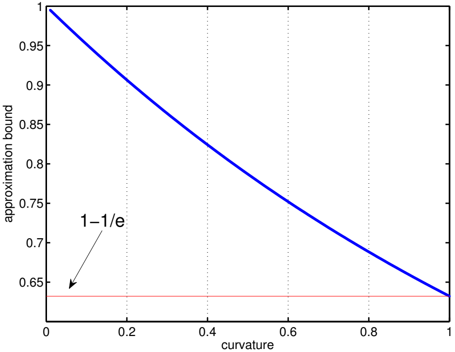
- For k-uniform matroid (i.e., k-cardinality constraints), then approximation factor becomes $\frac{1}{c}(1 - e^{-c})$



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- For purely supermodular optimization (i.e., $\kappa_f = 0$) we get that greedy has a guarantee of $1 - \kappa_g$.

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- Combinations of the above (e.g., knapsack & multiple matroid constraints).

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Given a polymatroid function f, and set of matroids $\{M_j = (E, \mathcal{I}_j)\}_{j=1}^p$, the above greedy algorithm returns sets S_i such that for each i we have $f(S_i) \geq \frac{1}{p+1} \max_{|S| \leq i, S \in \bigcap_{i=1}^p \mathcal{I}_i} f(S)$, assuming such sets exists.

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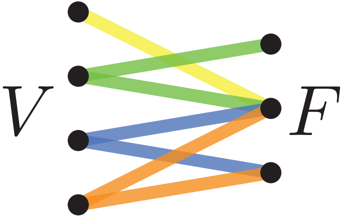

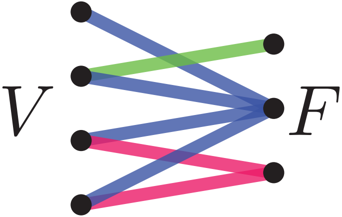

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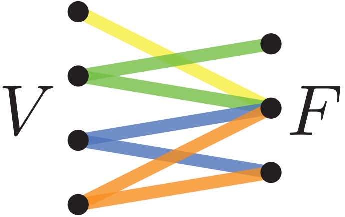

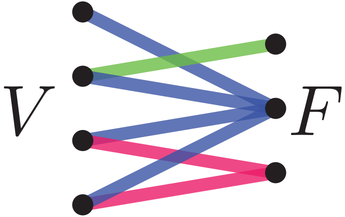

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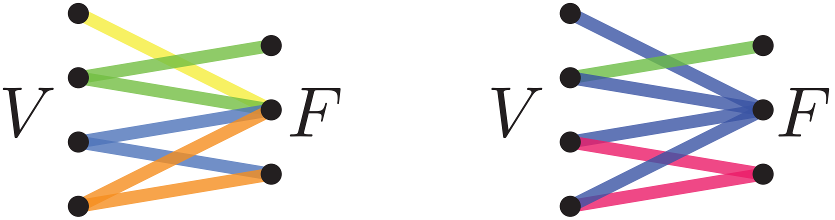

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- In bipartite graph case, therefore, can be solved in polynomial time.

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- Then a Hamiltonian cycle exists iff there is an n-element intersection of M_1, M_2, and M_3.

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- But bipartite graph example gives us hope for 2 matroids, as in that case we can easily solve $\max |X|$ s.t. $x \in \mathcal{I}_1 \cap \mathcal{I}_2$.

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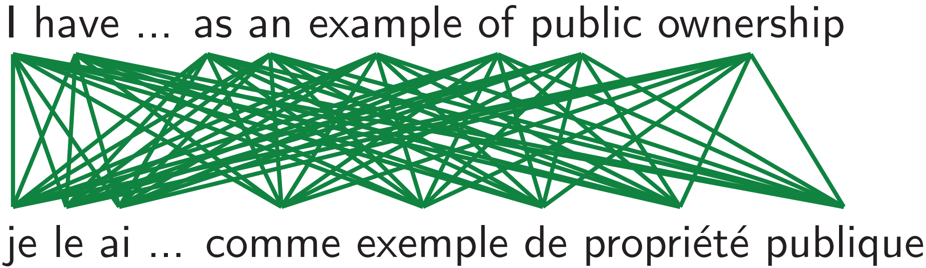
- E corresponds to, say, an English language sentence and F corresponds to a French language sentence — goal is to form a matching (an alignment) between the two.

Greedy over > 1 matroids: Multiple Language Alignment

- Consider English string and French string, set up as a bipartite graph.



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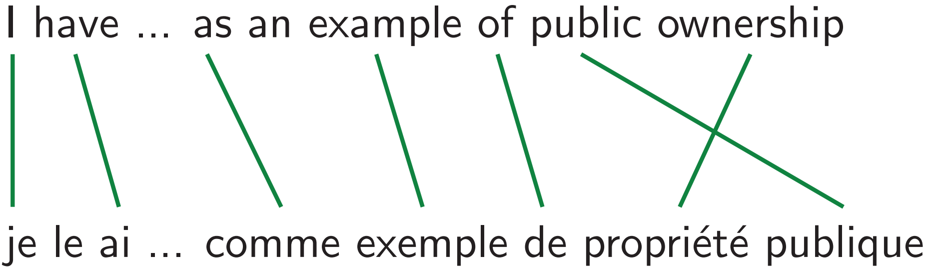
- One possible alignment, a matching, with score as sum of edge weights.



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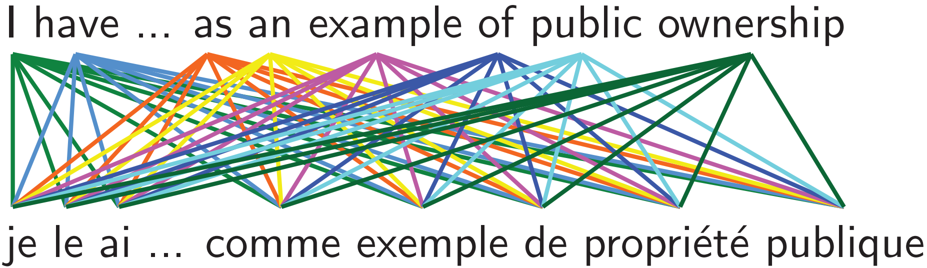

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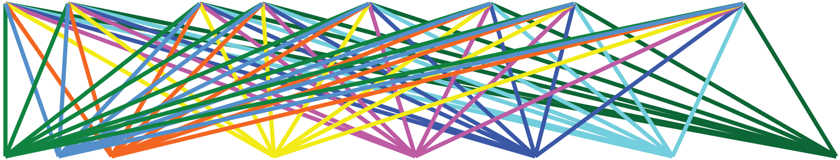

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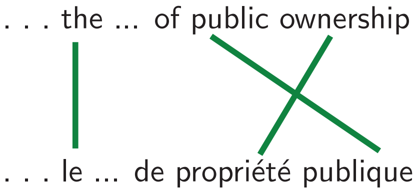

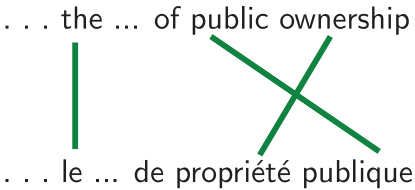

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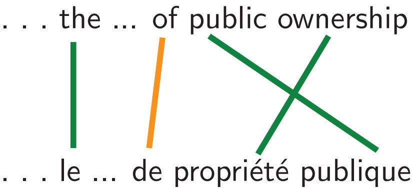

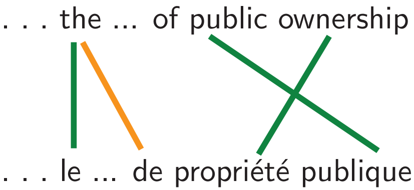

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- We can solve this via submodular maximization subject to multiple matroid independence constraints as we next describe ...

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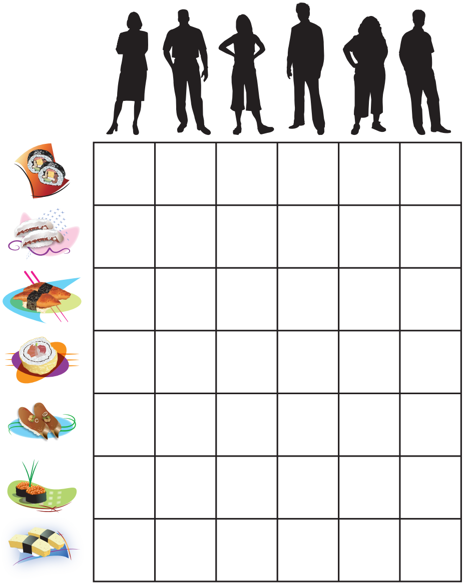

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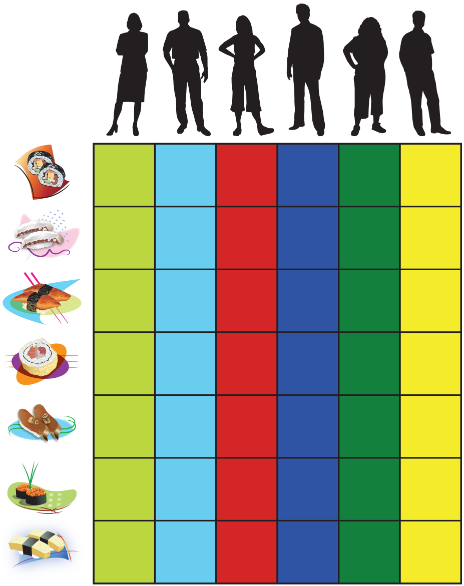

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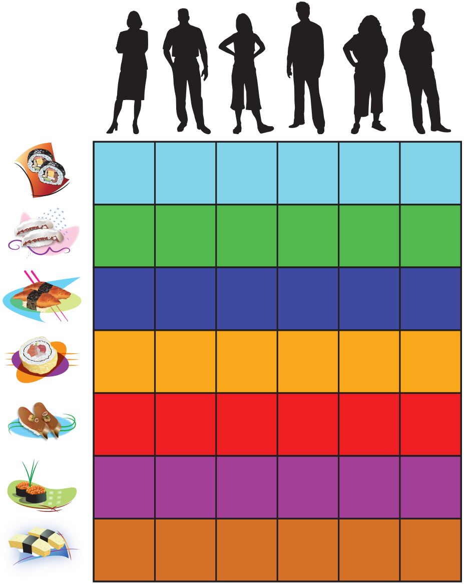

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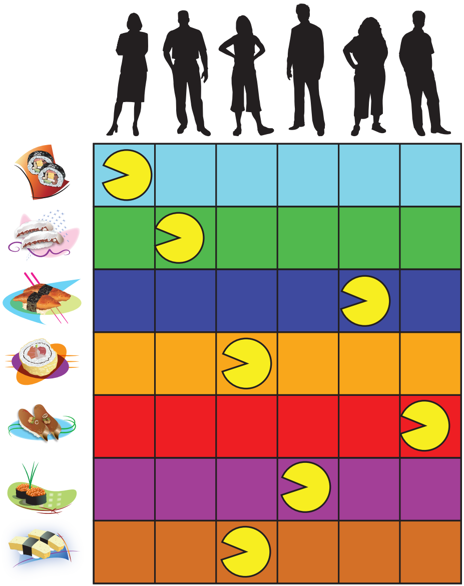

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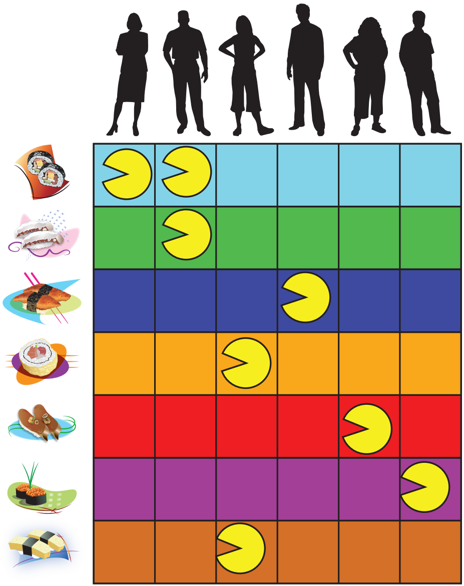

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- Important: A knapsack constraint yields an independence system (down closed) but it is not a matroid!
- $c(e)$ may be seen as the cost of item e and if $c(e) = 1$ for all e, then we recover the cardinality constraint we saw earlier.

Monotone Submodular over Knapsack Constraint

- Greedy can be seen as choosing the best gain: Starting with $S_0 = \emptyset$, we repeat the following greedy step

$$S_{i+1} = S_i \cup \left\{ \underset{v \in V \setminus S_i}{\operatorname{argmax}} \Big(f(S_i \cup \{v\}) - f(S_i) \Big) \right\} \tag{13.30}$$

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- Core idea in knapsack case: Greedy can be extended to choose next whatever looks cost-normalized best, i.e., Starting some initial set S_0, we repeat the following cost-normalized greedy step

$$S_{i+1} = S_i \cup \left\{ \operatorname*{argmax}_{v \in V \setminus S_i} \frac{f(S_i \cup \{v\}) - f(S_i)}{c(v)} \right\} \tag{13.31}$$

which we repeat until $c(S_{i+1}) > b$ and then take S_i as the solution.

A Knapsack Constraint

- There are a number of ways of getting approximation bounds using this strategy.
- If we run the normalized greedy procedure starting with $S_0 = \emptyset$, and compare the solution found with the max of the singletons $\max_{v \in V} f(\{v\})$, choosing the max, then we get a $(1 - e^{-1/2}) \approx 0.39$ approximation, in $O(n^2)$ time (Minoux trick also possible for further speed)
- Partial enumeration: On the other hand, we can get a $(1 - e^{-1}) \approx 0.63$ approximation in $O(n^5)$ time if we run the above procedure starting from all sets of cardinality three (so restart for all S_0 such that $|S_0| = 3$), and compare that with the best singleton and pairwise solution.
- Extending something similar to this to d simultaneous knapsack constraints is possible as well.

Local Search Algorithms

From J. Vondrak

- Local search involves switching up to t elements, as long as it provides a (non-trivial) improvement; can iterate in several phases. Some examples follow:
- 1/3 approximation to unconstrained non-monotone maximization [Feige, Mirrokni, Vondrak, 2007]
- $1/(k + 2 + \frac{1}{k} + \delta_t)$ approximation for non-monotone maximization subject to k matroids [Lee, Mirrokni, Nagarajan, Sviridenko, 2009]
- $1/(k + \delta_t)$ approximation for monotone submodular maximization subject to $k \geq 2$ matroids [Lee, Sviridenko, Vondrak, 2010].

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- Thus, any approximation algorithm must be for unipolar submodular functions. E.g., non-negative but otherwise arbitrary submodular functions.
- We may get a $(\frac{1}{3} - \frac{\epsilon}{n})$ approximation for maximizing non-monotone non-negative submodular functions, with most $O(\frac{1}{\epsilon} n^3 \log n)$ function calls using approximate local maxima.

Submodularity and local optima

- Given any submodular function f, a set $S \subseteq V$ is a local maximum of f if $f(S - v) \leq f(S)$ for all $v \in S$ and $f(S + v) \leq f(S)$ for all $v \in V \setminus S$ (i.e., local in a Hamming ball of radius 1).

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- Similarly, given $v_1, v_2 \notin S$, and $f(S + v_1) \leq f(S)$ and $f(S + v_2) \leq f(S)$. Submodularity requires $f(S + v_1) + f(S + v_2) \geq f(S) + f(S + v_1 + v_2)$ which requires $f(S + v_1 + v_2) \leq f(S)$.

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- This is the approach that yields the $(\frac{1}{3} - \frac{\epsilon}{n})$ approximation algorithm.

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Algorithm 6: Randomized Linear-time non-monotone submodular max

```
Set L ← ∅ ; U ← V     /* Lower L, upper U. Invariant: L ⊆ U */ ;
Order elements of V = (v_1, v_2, ..., v_n) arbitrarily ;
for i ← 0 ... |V| do
    a ← [f(v_i|L)]_+; b ← [−f(U|U \ {v_i})]_+ ;
    if a = b = 0 then p ← 1/2 ;
    ;
    else p ← a/(a + b);
    ;
    if Flip of coin with Pr(heads) = p draws heads then
        L ← L ∪ {v_i} ;
    Otherwise /* if the coin drew tails, an event with prob. 1 − p */
        U ← U \ {v}
return L (which is the same as U at this point)
```

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- It may be possible to choose the random order smartly to get better results in practice.

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- Often the computational costs of the algorithms are prohibitive (e.g., exponential in k) with large constants, so these algorithms might not scale.
- On the other hand, these algorithms offer deep and interesting intuition into submodular functions, beyond what we have covered here.

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- We can recover $1 - 1/e$ approximation using the continuous greedy algorithm on the multilinear extension and then using pipage rounding to re-integerize the solution (see J. Vondrak's publications).
- More general constraints are possible too, as we see on the next table (for references, see Jan Vondrak's publications `http://theory.stanford.edu/~jvondrak/`).

Submodular Max Summary - 2012: From J. Vondrak

Monotone Maximization

Constraint	Approximation	Hardness	Technique
$\|S\| \leq k$	$1 - 1/e$	$1 - 1/e$	greedy
matroid	$1 - 1/e$	$1 - 1/e$	multilinear ext.
$O(1)$ knapsacks	$1 - 1/e$	$1 - 1/e$	multilinear ext.
k matroids	$k + \epsilon$	$k/\log k$	local search
k matroids and $O(1)$ knapsacks	$O(k)$	$k/\log k$	multilinear ext.

Nonmonotone Maximization

Constraint	Approximation	Hardness	Technique
Unconstrained	$1/2$	$1/2$	combinatorial
matroid	$1/e$	0.48	multilinear ext.
$O(1)$ knapsacks	$1/e$	0.49	multilinear ext.
k matroids	$k + O(1)$	$k/\log k$	local search
k matroids and $O(1)$ knapsacks	$O(k)$	$k/\log k$	multilinear ext.