Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture 13 —

http://www.ee.washington.edu/people/faculty/bilmes/classes/ee563_spring_2018/

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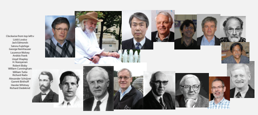
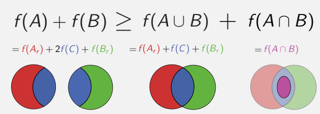
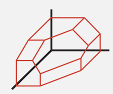
May 9th, 2018







Cumulative Outstanding Reading

- Read chapter 1 from Fujishige's book.
- Read chapter 2 from Fujishige's book.
- Read chapter 3 from Fujishige's book.
- Read chapter 4 from Fujishige's book.

Announcements, Assignments, and Reminders

- Next homework is posted on canvas. Due Thursday 5/10, 11:59pm.
- As always, if you have any questions about anything, please ask then via our discussion board (https://canvas.uw.edu/courses/1216339/discussion_topics). Can meet at odd hours via zoom (send message on canvas to schedule time to chat).

Class Road Map - EE563

- L1(3/26): Motivation, Applications, & Basic Definitions,
- L2(3/28): Machine Learning Apps (diversity, complexity, parameter, learning target, surrogate).
- L3(4/2): Info theory exs, more apps, definitions, graph/combinatorial examples
- L4(4/4): Graph and Combinatorial Examples, Matrix Rank, Examples and Properties, visualizations
- L5(4/9): More Examples/Properties/ Other Submodular Defs., Independence,
- L6(4/11): Matroids, Matroid Examples, Matroid Rank, Partition/Laminar Matroids
- L7(4/16): Laminar Matroids, System of Distinct Reps, Transversals, Transversal Matroid, Matroid Representation, Dual Matroids
- L8(4/18): Dual Matroids, Other Matroid Properties, Combinatorial Geometries, Matroids and Greedy.
- L9(4/23): Polyhedra, Matroid Polytopes, Matroids $\to$ Polymatroids
- L10(4/29): Matroids $\to$ Polymatroids, Polymatroids, Polymatroids and Greedy,
- L11(4/30): Polymatroids, Polymatroids and Greedy
- L12(5/2): Polymatroids and Greedy, Extreme Points, Cardinality Constrained Maximization
- L13(5/7): Constrained Submodular Maximization
- L14(5/9):
- L15(5/14):
- L16(5/16):
- L17(5/21):
- L18(5/23):
- L–(5/28): Memorial Day (holiday)
- L19(5/30):
- L21(6/4): Final Presentations maximization.

Last day of instruction, June 1st. Finals Week: June 2-8, 2018.

Multiple Polytopes associated with arbitrary f

- Given an arbitrary submodular function $f : 2^V \to R$ (not necessarily a polymatroid function, so it need not be positive, monotone, etc.).
- If $f(\emptyset) \neq 0$, can set $f'(A) = f(A) - f(\emptyset)$ without destroying submodularity. This does not change any minima, (i.e., $\text{argmin}_A f(A) = \text{argmin}_{A'} f'(A)$) so we often assume all functions are normalized $f(\emptyset) = 0$.
  Note that due to constraint $x(\emptyset) \leq f(\emptyset)$, we must have $f(\emptyset) \geq 0$ since if not (i.e., if $f(\emptyset) < 0$), then P_f^+ doesn't exist.
  Another form of normalization can do is:

$$f'(A) = \begin{cases} f(A) & \text{if } A \neq \emptyset \\ 0 & \text{if } A = \emptyset \end{cases} \tag{13.1}$$

  This preserves submodularity due to $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$, and if $A \cap B = \emptyset$ then r.h.s. only gets smaller when $f(\emptyset) \geq 0$.
- We can define several polytopes:

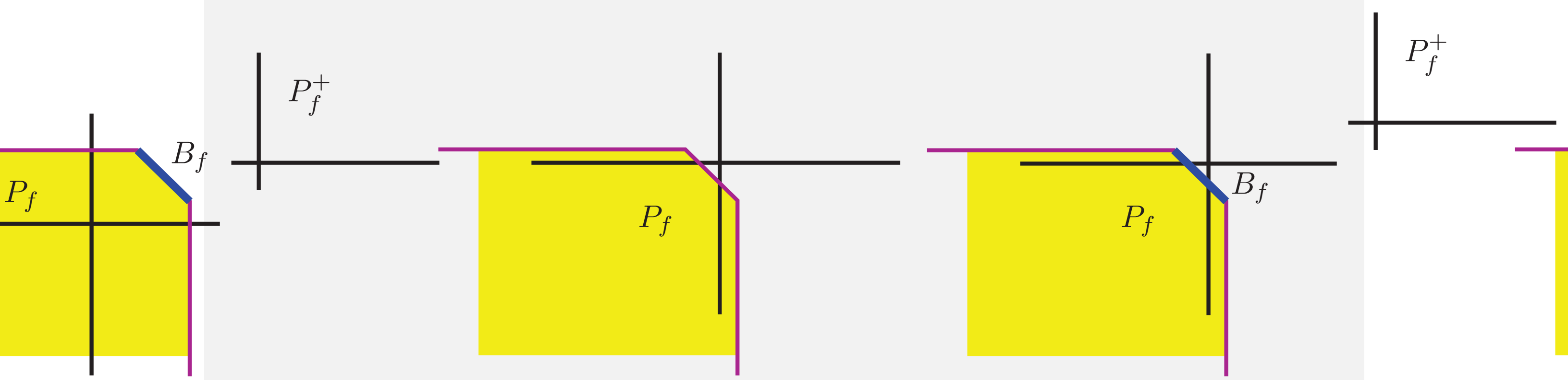
$$P_f = \{x \in \mathbb{R}^E : x(S) \leq f(S), \forall S \subseteq E\} \tag{13.2}$$

$$P_f^+ = P_f \cap \{x \in \mathbb{R}^E : x \geq 0\} \tag{13.3}$$

$$B_f = P_f \cap \{x \in \mathbb{R}^E : x(E) = f(E)\} \tag{13.4}$$

- P_f is what is sometimes called the extended polytope (sometimes notated as EP_f.

Multiple Polytopes in 2D associated with f



$$P_f^+ = P_f \cap \{x \in \mathbb{R}^E : x \geq 0\} \tag{13.1}$$

$$P_f = \{x \in \mathbb{R}^E : x(S) \leq f(S), \forall S \subseteq E\} \tag{13.2}$$

$$B_f = P_f \cap \{x \in \mathbb{R}^E : x(E) = f(E)\} \tag{13.3}$$

A polymatroid function's polyhedron is a polymatroid.

Theorem 13.2.1

Let f be a submodular function defined on subsets of E. For any $x \in \mathbb{R}^E$, we have:

$$\text{rank}(x) = \max\left(y(E) : y \le x, y \in P_f\right) = \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \quad (13.1)$$

Essentially the same theorem as Theorem **??**, but note P_f rather than P_f^+. Taking $x = 0$ we get:

Corollary 13.2.2

Let f be a submodular function defined on subsets of E. We have:

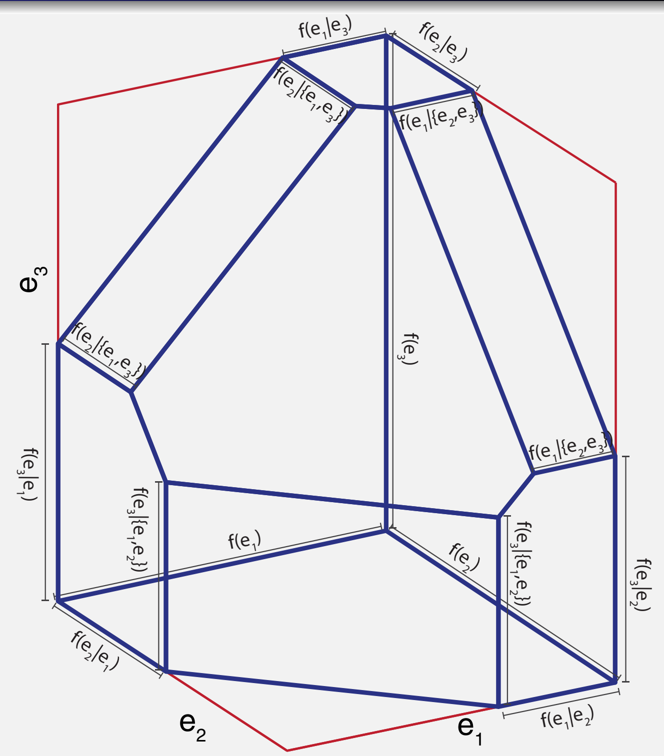
$$\text{rank}(0) = \max\left(y(E) : y \le 0, y \in P_f\right) = \min\left(f(A) : A \subseteq E\right) \quad (13.2)$$

Polymatroid extreme points

Polymatroid extreme points

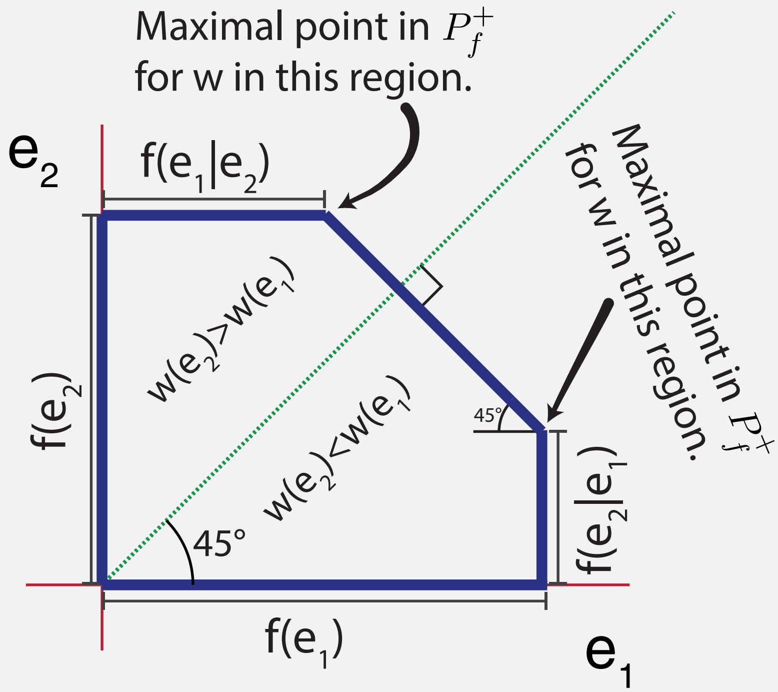
Polymatroid with labeled edge lengths

- Recall $f(e|A) = f(A+e) - f(A)$
- Notice how submodularity, $f(e|B) \leq f(e|A)$ for $A \subseteq B$, defines the shape of the polytope.
- In fact, we have strictness here $f(e|B) < f(e|A)$ for $A \subset B$.
- Also, consider how the greedy algorithm proceeds along the edges of the polytope.



Intuition: why greedy works with polymatroids

- Given w, the goal is to find $x = (x(e_1), x(e_2))$ that maximizes $x^\intercal w = x(e_1)w(e_1) + x(e_2)w(e_2)$.
- If $w(e_2) > w(e_1)$ the upper extreme point indicated maximizes $x^\intercal w$ over $x \in P_f^+$.
- If $w(e_2) < w(e_1)$ the lower extreme point indicated maximizes $x^\intercal w$ over $x \in P_f^+$.



The Greedy Algorithm for Submodular Max

A bit more precisely:

Algorithm 1: The Greedy Algorithm

Set $S_0 \leftarrow \emptyset$;
for $i \leftarrow 0 \ldots |E| - 1$ **do**
  Choose v_i as follows:
  $v_i \in \operatorname{argmax}_{v \in V \setminus S_i} f(\{v\}|S_i) = \operatorname{argmax}_{v \in V \setminus S_i} f(S_i \cup \{v\})$;
  Set $S_{i+1} \leftarrow S_i \cup \{v_i\}$;

Greedy Algorithm for Card. Constrained Submodular Max

- This algorithm has a guarantee

Theorem 13.2.1

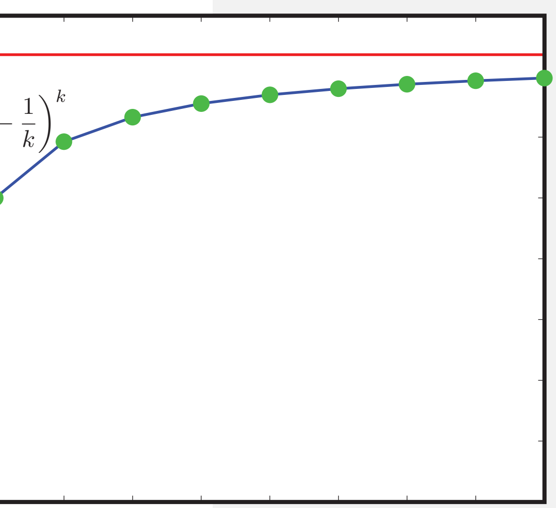
Given a polymatroid function f , the above greedy algorithm returns sets S_i such that for each i we have $f(S_i) \geq (1 - 1/e) \max_{|S| \leq i} f(S)$.

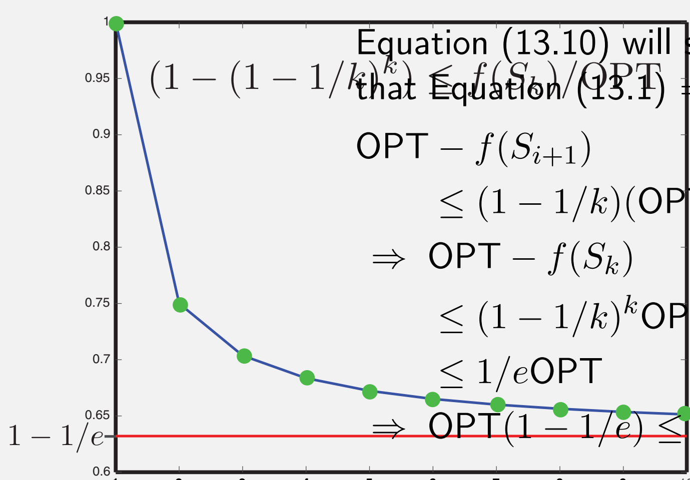
- To approximately find $A^* \in \operatorname{argmax} \{f(A) : |A| \leq k\}$, we repeat the greedy step until $k = i + 1$:
- Again, since this generalizes max k-cover, Feige (1998) showed that this can't be improved. Unless $P = NP$, no polynomial time algorithm can do better than $(1 - 1/e + \epsilon)$ for any $\epsilon > 0$.

The Greedy Algorithm: $1 - 1/e$ intuition.

- At step $i < k$, greedy chooses v_i to maximize $f(v|S_i)$.
- Let S^* be optimal solution (of size k) and OPT $= f(S^*)$. By submodularity, we will show:

$$\exists v \in V \setminus S_i : f(v|S_i) = f(S_i + v|S_i) \geq \frac{1}{k}(\text{OPT} - f(S_i)) \qquad (13.1)$$





Equation (13.10) will show that Equation (13.1) $\Rightarrow$:

$$\begin{aligned} &\text{OPT} - f(S_{i+1}) \\ &\quad \leq (1 - 1/k)(\text{OPT} - f(S_i)) \\ &\Rightarrow \text{OPT} - f(S_k) \\ &\quad \leq (1 - 1/k)^k \text{OPT} \\ &\quad \leq 1/e\text{OPT} \\ &\Rightarrow \text{OPT}(1 - 1/e) \leq f(S_k) \end{aligned}$$

Cardinality Constrained Polymatroid Max Theorem

Theorem 13.3.1 (Nemhauser et al. 1978)

Given non-negative monotone submodular function $f : 2^V \to \mathbb{R}_+$, define $\{S_i\}_{i \geq 0}$ to be the chain formed by the greedy algorithm (Eqn. (??)). Then for all $k, \ell \in \mathbb{Z}_{++}$, we have:

$$f(S_\ell) \geq (1 - e^{-\ell/k}) \max_{S:|S| \leq k} f(S) \tag{13.2}$$

and in particular, for $\ell = k$, we have $f(S_k) \geq (1 - 1/e) \max_{S:|S| \leq k} f(S)$.

- k is size of optimal set, i.e., OPT $= f(S^*)$ with $|S^*| = k$
- ℓ is size of set we are choosing (i.e., we choose S_ℓ from greedy chain).
- Bound is how well does S_ℓ (of size ℓ) do relative to S^*, the optimal set of size k.
- Intuitively, bound should get worse when $\ell < k$ and get better when $\ell > k$.

Cardinality Constrained Polymatroid Max Theorem

Proof of Theorem 13.3.1.

- Fix ℓ (number of items greedy will chose) and k (size of optimal set to compare against).
- Set $S^* \in \operatorname{argmax} \{f(S) : |S| \leq k\}$
- w.l.o.g. assume $|S^*| = k$.
- Order $S^* = (v_1^*, v_2^*, \ldots, v_k^*)$ arbitrarily.
- Let $S_i = (v_1, v_2, \ldots, v_i)$ be the greedy order chain chosen by the algorithm, for $i \in \{1, 2, \ldots, \ell\}$.
- Then the following inequalities (on the next slide) follow:

...

Cardinality Constrained Polymatroid Max Theorem

... proof of Theorem 13.3.1 cont.

- For all $i < \ell$, we have

$$f(S^*) \leq f(S^* \cup S_i) = f(S_i) + f(S^*|S_i) \tag{13.3}$$
$$= f(S_i) + \sum_{j=1}^{k} f(v_j^*|S_i \cup \{v_1^*, v_2^*, \ldots, v_{j-1}^*\}) \tag{13.4}$$
$$\leq f(S_i) + \sum_{v \in S^*} f(v|S_i) \tag{13.5}$$
$$\leq f(S_i) + \sum_{v \in S^*} f(v_{i+1}|S_i) = f(S_i) + \sum_{v \in S^*} f(S_{i+1}|S_i) \tag{13.6}$$
$$= f(S_i) + kf(S_{i+1}|S_i) \tag{13.7}$$

- Therefore, we have Equation 13.1, i.e.,:

$$f(S^*) - f(S_i) \leq kf(S_{i+1}|S_i) = k(f(S_{i+1}) - f(S_i)) \tag{13.8}$$

...

Cardinality Constrained Polymatroid Max Theorem

... proof of Theorem 13.3.1 cont.

- Define gap $\delta_i \triangleq f(S^*) - f(S_i)$, so $\delta_i - \delta_{i+1} = f(S_{i+1}) - f(S_i)$, giving

$$\delta_i \leq k(\delta_i - \delta_{i+1}) \tag{13.9}$$

or

$$\delta_{i+1} \leq (1 - \frac{1}{k})\delta_i \tag{13.10}$$

- The relationship between δ_0 and δ_ℓ is then

$$\delta_l \leq (1 - \frac{1}{k})^\ell \delta_0 \tag{13.11}$$

- Now, $\delta_0 = f(S^*) - f(\emptyset) \leq f(S^*)$ since $f \geq 0$.
- Also, by variational bound $1 - x \leq e^{-x}$ for $x \in \mathbb{R}$, we have

$$\delta_\ell \leq (1 - \frac{1}{k})^\ell \delta_0 \leq e^{-\ell/k} f(S^*) \tag{13.12}$$

...

Cardinality Constrained Polymatroid Max Theorem

. . . proof of Theorem 13.3.1 cont.

- When we identify $\delta_l = f(S^*) - f(S_\ell)$, a bit of rearranging then gives:

$$f(S_\ell) \geq (1 - e^{-\ell/k}) f(S^*) \tag{13.13}$$

□

- With $\ell = k$, when picking k items, greedy gets $(1 - 1/e) \approx 0.6321$ bound. This means that if S_k is greedy solution of size k, and S^* is an optimal solution of size k, $f(S_k) \geq (1 - 1/e) f(S^*) \approx 0.6321 f(S^*)$.
- What if we want to guarantee a solution no worse than $.95 f(S^*)$ where $|S^*| = k$? Set $0.95 = (1 - e^{-\ell/k})$, which gives $\ell = \lceil -k \ln(1 - 0.95) \rceil = 4k$. And $\lceil -\ln(1 - 0.999) \rceil = 7$.
- So solution, in the worst case, quickly gets very good. Typical/practical case is much better.

Greedy running time

- Greedy computes a new maximum $n = |V|$ times, and each maximum computation requires $O(n)$ comparisons, leading to $O(n^2)$ computation for greedy.
- This is the best we can do for arbitrary functions, but $O(n^2)$ is not practical to some.
- Greedy can be made much faster in practice by a simple strategy made possible, once again, via the use of submodularity.
- This is called Minoux's 1977 Accelerated Greedy strategy (and has been rediscovered a few times, e.g., "Lazy greedy"), and runs much faster while still producing same answer.
- We describe it next:

Minoux's Accelerated Greedy for Submodular Functions

- At stage i in the algorithm, we have a set of gains $f(v|S_i)$ for all $v \notin S_i$. Store these values $\alpha_v \leftarrow f(v|S_i)$ in sorted priority queue.
- Priority queue, $O(1)$ to find max, $O(\log n)$ to insert in right place.
- Once we choose a max v, then set $S_{i+1} \leftarrow S_i + v$.
- For $v \notin S_{i+1}$ we have $f(v|S_{i+1}) \leq f(v|S_i)$ by submodularity.
- Therefore, if we find a v' such that $f(v'|S_{i+1}) \geq \alpha_v$ for all $v \neq v'$, then since

$$f(v'|S_{i+1}) \geq \alpha_v = f(v|S_i) \geq f(v|S_{i+1}) \quad (13.14)$$

  we have the true max, and we need not re-evaluate gains of other elements again.
- Strategy is: find the $\text{argmax}_{v' \in V \setminus S_{i+1}} \alpha_{v'}$, and then compute the real $f(v'|S_{i+1})$. If it is greater than all other α_v's then that's the next greedy step. Otherwise, replace $\alpha_{v'}$ with its real value, resort ($O(\log n)$), and repeat.

Minoux's Accelerated Greedy for Submodular Functions

- Minoux's algorithm is exact, in that it has the same guarantees as does the standard $O(n^2)$ greedy algorithm (will return the same answers, i.e., those having the $1 - 1/e$ guarantee).
- In practice: Minoux's trick has enormous speedups ($\approx 700\times$) over the standard greedy procedure due to reduced function evaluations and use of good data structures (priority queue).
- When choosing a of size k, naïve greedy algorithm is $O(nk)$ but accelerated variant at the very best does $O(n + k)$, so this limits the speedup.
- Algorithm has been rediscovered (I think) independently (CELF - cost-effective lazy forward selection, Leskovec et al., 2007)
- Can be used used for "big data" sets (e.g., social networks, selecting blogs of greatest influence, document summarization, etc.).

Priority Queue

- Use a priority queue Q as a data structure: operations include:
  - Insert an item (v, α) into queue, with $v \in V$ and $\alpha \in \mathbb{R}$.
  $$\text{insert}(Q, (v, \alpha)) \tag{13.15}$$
  - Pop the item (v, α) with maximum value α off the queue.
  $$(v, \alpha) \leftarrow \text{pop}(Q) \tag{13.16}$$
  - Query the value of the max item in the queue
  $$\max(Q) \in \mathbb{R} \tag{13.17}$$
- On next slide, we call a popped item "fresh" if the value (v, α) popped has the correct value $\alpha = f(v|S_i)$. Use extra "bit" to store this info
- If a popped item is fresh, it must be the maximum — this can happen if, at given iteration, v was first popped and neither fresh nor maximum so placed back in the queue, and it then percolates back to the top at which point it is fresh — thereby avoid extra queue check.

Minoux's Accelerated Greedy Algorithm Submodular Max

Algorithm 2: Minoux's Accelerated Greedy Algorithm

```
1  Set S_0 ← ∅ ; i ← 0 ; Initialize priority queue Q ;
2  for v ∈ E do
3      INSERT(Q, f(v))
4  repeat
5      (v, α) ← pop(Q) ;
6      if α not "fresh" then
7          recompute α ← f(v|S_i)
8      if (popped α in line 5 was "fresh") OR (α ≥ max(Q)) then
9          Set S_{i+1} ← S_i ∪ {v} ;
10         i ← i + 1 ;
11     else
12         insert(Q, (v, α))
13 until i = |E|;
```

(Minimum) Submodular Set Cover

- Given polymatroid f, goal is to find a covering set of minimum cost:

$$S^* \in \operatorname*{argmin}_{S \subseteq V} |S| \text{ such that } f(S) \geq \alpha \tag{13.18}$$

where α is a "cover" requirement.

- Normally take $\alpha = f(V)$ but defining $f'(A) = \min\{f(A), \alpha\}$ we can take any α. Hence, we have equivalent formulation:

$$S^* \in \operatorname*{argmin}_{S \subseteq V} |S| \text{ such that } f'(S) \geq f'(V) \tag{13.19}$$

- Note that this immediately generalizes standard set cover, in which case $f(A)$ is the cardinality of the union of sets indexed by A.
- Greedy Algorithm: Pick the first chain item S_i chosen by aforementioned greedy algorithm such that $f(S_i) \geq \alpha$ and output that as solution.

Polymatroids, Greedy, and Cardinality Constrained Maximization | Curvature | Submodular Max w. Other Constraints | Submodular Max w. Other Constraints

(Minimum) Submodular Set Cover: Approximation Analysis

- For integer valued f, this greedy algorithm an $O(\log(\max_{s \in V} f(\{s\})))$ approximation. Let S^* be optimal, and S^{G} be greedy solution, then

$$|S^{\mathsf{G}}| \leq |S^*| H(\max_{s \in V} f(\{s\})) = |S^*| O(\log_e(\max_{s \in V} f(\{s\}))) \tag{13.20}$$

where H is the harmonic function, i.e., $H(d) = \sum_{i=1}^{d}(1/i)$.

- If f is not integral value, then bounds we get are of the form:

$$|S^{\mathsf{G}}| \leq |S^*| \left(1 + \log_e \frac{f(V)}{f(V) - f(S_{T-1})}\right) \tag{13.21}$$

wehre S_T is the final greedy solution that occurs at step T.

- Set cover is hard to approximate with a factor better than $(1-\epsilon)\log \alpha$, where α is the desired cover constraint.

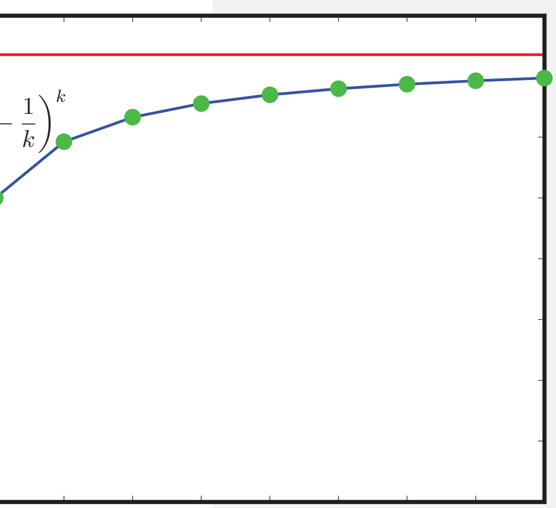
Summary: Monotone Submodular Maximization

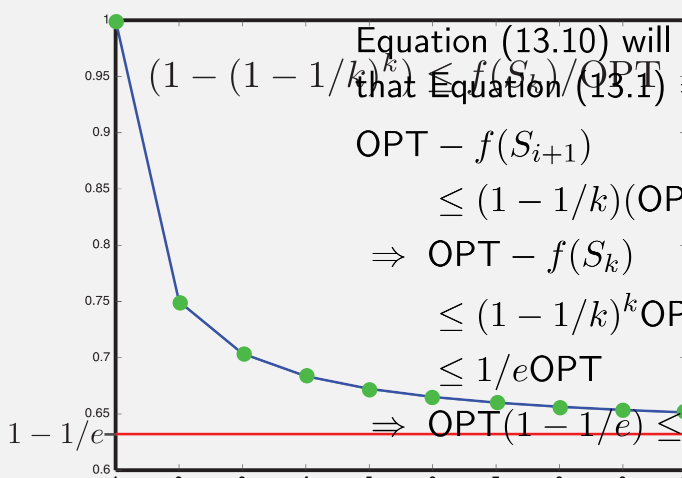
- Only makes sense when there is a constraint.
- We discussed cardinality constraint
- Generalizes the max k-cover problem, and also similar to the set cover problem.
- Simple greedy algorithm gets $1 - e^{-\ell/k}$ approximation, where k is size of optimal set we compare against, and ℓ is size of set greedy algorithm chooses.
- Submodular cover: min. $|S|$ s.t. $f(S) \geq \alpha$.
- Minoux's accelerated greedy trick.

The Greedy Algorithm: $1 - 1/e$ intuition.

- At step $i < k$, greedy chooses v_i to maximize $f(v|S_i)$.
- Let S^* be optimal solution (of size k) and OPT $= f(S^*)$. By submodularity, we will show:

$$\exists v \in V \setminus S_i : f(v|S_i) = f(S_i + v|S_i) \geq \frac{1}{k}(\text{OPT} - f(S_i)) \qquad (13.1)$$





Equation (13.10) will show that Equation (13.1) $\Rightarrow$:

$$\begin{aligned}\text{OPT} - f(S_{i+1}) &\leq (1 - 1/k)(\text{OPT} - f(S_i)) \\ \Rightarrow \text{OPT} - f(S_k) &\leq (1-1/k)^k \text{OPT} \\ &\leq 1/e\text{OPT} \\ \Rightarrow \text{OPT}(1 - 1/e) &\leq f(S_k)\end{aligned}$$

Randomized greedy

- How can we produce a randomized greedy strategy, one where each greedy sweep produces a set that, on average, has a $1 - 1/e$ guarantee?
- Suppose the following holds:

$$E[f(a_{i+1}|A_i)] \geq \frac{f(OPT) - f(A_i)}{k} \tag{13.22}$$

where $A_i = (a_1, a_2, \dots, a_i)$ are the first i elements chosen by the strategy.

Curvature of a Submodular function

- For any submodular function, we have $f(j|S) \leq f(j|\emptyset)$ so that $f(j|S)/f(j|\emptyset) \leq 1$ whenever $f(j|\emptyset) \neq 0$.
- For $f : 2^V \to \mathbb{R}_+$ (non-negative) functions, we also have $f(j|S)/f(j|\emptyset) \geq 0$ — and $= 0$ whenever j is "spanned" by S.
- The total curvature of a submodular function is defined as follows:

$$c \triangleq 1 - \min_{S, j \notin S : f(j|\emptyset) \neq 0} \frac{f(j|S)}{f(j|\emptyset)} = 1 - \min_{f(j) \neq 0} \frac{f(j|V \setminus j)}{f(j)} \tag{13.23}$$

- $c \in [0,1]$. When $c = 0$, $f(j|S) = f(j|\emptyset)$ for all S, j, a sufficient condition for modularity, and we saw in Theorem **??** that greedy is optimal for max weight indep. set of a matroid.
- For f with curvature c, then $\forall A \subseteq V$, $\forall v \notin a$, $\forall c' \geq c$:

$$f(A + v) - f(A) \geq (1 - c')f(v) \tag{13.24}$$

$$f(v) \geq f(v|A) = f(v)\frac{f(v|A)}{f(v)} \geq f(v) \min_{v'} \frac{f(v'|A)}{f(v')} = (1-c)f(v) \geq (1-c')f(v) \tag{13.25}$$

- When $c = 1$ then submodular function is "maximally curved", i.e., there exists is a subset that fully spans some other element.

Curvature of a Submodular function

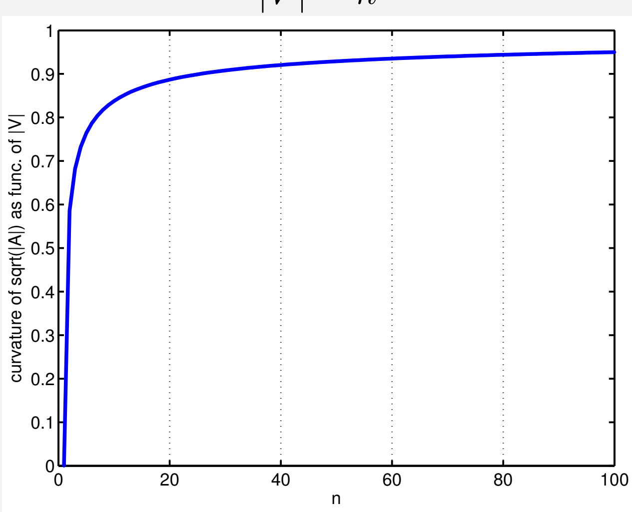
- By submodularity, total curvature can be computed in either form:

$$c \triangleq 1 - \min_{S, j \notin S : f(j|\emptyset) \neq 0} \frac{f(j|S)}{f(j|\emptyset)} = 1 - \min_{j : f(j|\emptyset) \neq 0} \frac{f(j|V \setminus \{j\})}{f(j|\emptyset)} \quad (13.26)$$

- Note: Matroid rank is either modular $c = 0$ or maximally curved $c = 1$ — hence, matroid rank can have only the extreme points of curvature, namely 0 or 1.
- Polymatroid functions are, in this sense, more nuanced, in that they allow non-extreme curvature, with $c \in [0, 1]$.
- It will be remembered the notion of "partial dependence" within polymatroid functions.

Curvature for $f(S) = \sqrt{|S|}$

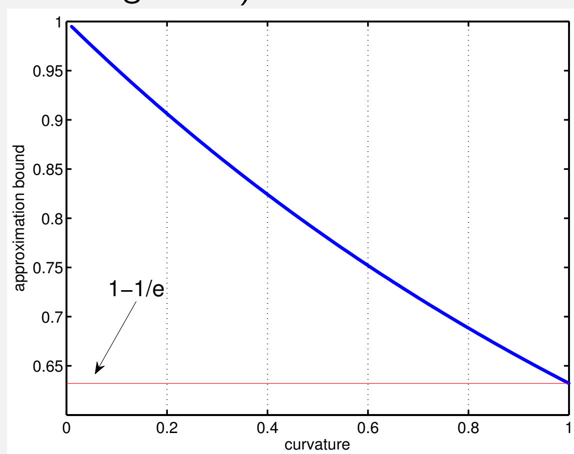
Curvature of $f(S) = \sqrt{|S|}$ as function of $|V| = n$



- $f(S) = \sqrt{|S|}$ with $|V| = n$ has curvature $1 - (\sqrt{n} - \sqrt{n-1})$.
- Approximation gets worse with bigger ground set.
- Functions of the form $f(S) = \sqrt{m(S)}$ where $m : V \to \mathbb{R}_+$, approximation worse with n if $\min_{i,j} |m(i) - m(j)|$ has a fixed lower bound with increasing n.

Curvature and approximation

- Curvature limitation can help the greedy algorithm in terms of approximation bounds.
- Conforti & Cornuéjols showed that greedy gives a $1/(1+c)$ approximation to $\max\{f(S) : S \in \mathcal{I}\}$ when f has total curvature c.
- Hence, greedy subject to matroid constraint is a $\max(1/(1+c), 1/2)$ approximation algorithm, and if $c < 1$ then it is better than 1/2 (e.g., with $c = 1/4$ then we have a 0.8 algorithm).
- For k-uniform matroid (i.e., k-cardinality constraints), then approximation factor becomes $\frac{1}{c}(1 - e^{-c})$



Submodular and Supermodular Curvature Approximation

- Let f be a polymatroid function and let g be a non-negative monotone non-decreasing supermodular function (e.g., $g(A) = \phi(m(A))$ where $\phi()$ is non-decreasing convex).
- Let $\kappa_f = 1 - \min_v \frac{f(v|V\setminus\{v\})}{f(v)}$ be the total submodular total curvature,
- Define $\kappa^g = 1 - \min_v \frac{g(v)}{g(v|V\setminus\{v\})}$ as a "supermodular curvature"
- $\kappa^g \in [0, 1]$ and $\kappa^g = 0$ means g is modular, $\kappa^g = 1$ means g is "fully curved"
- Form function $h(A) = f(A) + g(A)$, then h is neither suBmodular nor suPermodular, and is known as a BP-function.
- Then the greedy algorithm on h has a guarantee of: $\frac{1}{\kappa_f}(1 - e^{-(1-\kappa_g)\kappa_f})$.
- For purely supermodular optimization (i.e., $\kappa_f = 0$) we get that greedy has a guarantee of $1 - \kappa_g$.

Generalizations

- Consider a k-uniform matroid $\mathcal{M} = (V, \mathcal{I})$ where $\mathcal{I} = \{S \subseteq V : |S| \leq k\}$, and consider problem $\max\{f(A) : A \in \mathcal{I}\}$
- Hence, the greedy algorithm is $1 - 1/e$ optimal for maximizing polymatroidal f subject to a k-uniform matroid constraint.
- Might be useful to allow an arbitrary matroid (e.g., partition matroid $\mathcal{I} = \{X \subseteq V : |X \cap V_i| \leq k_i \text{ for all } i = 1, \ldots, \ell\}$., or a transversal, etc).
- Knapsack constraint: if each item $v \in V$ has a cost $c(v)$, we may ask for $c(S) \leq b$ where b is a budget, in units of costs. Q: Is $\mathcal{I} = \{I : c(I) \leq b\}$ the independent sets of a matroid?
- We may wish to maximize f subject to multiple matroid constraints. I.e., $S \in \mathcal{I}_1, S \in \mathcal{I}_2, \ldots, S \in \mathcal{I}_p$ where $\mathcal{I}_i$ are independent sets of the i^{th} matroid.
- Combinations of the above (e.g., knapsack & multiple matroid constraints).

Greedy over multiple matroids

- Obvious heuristic is to use the greedy step but always stay feasible.
- I.e., Starting with $S_0 = \emptyset$, we repeat the following greedy step

$$S_{i+1} = S_i \cup \left\{ \underset{v \in V \setminus S_i \,:\, S_i + v \in \bigcap_{i=1}^p \mathcal{I}_i}{\operatorname{argmax}} f(S_i \cup \{v\}) \right\} \tag{13.27}$$

- That is, we keep choosing next whatever feasible element looks best.
- This algorithm is simple and also has a guarantee

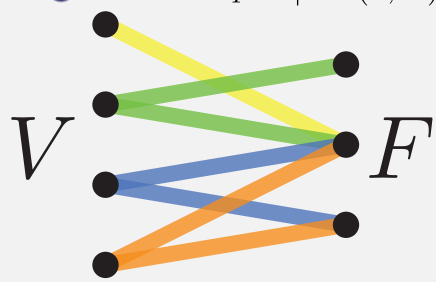
Theorem 13.5.1

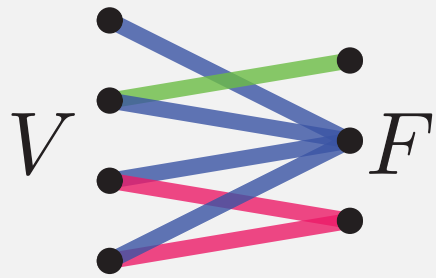
Given a polymatroid function f, and set of matroids $\{M_j = (E, \mathcal{I}_j)\}_{j=1}^p$, the above greedy algorithm returns sets S_i such that for each i we have $f(S_i) \geq \frac{1}{p+1} \max_{|S| \leq i, S \in \bigcap_{i=1}^p \mathcal{I}_i} f(S)$, assuming such sets exists.

- For one matroid, we have a 1/2 approximation.
- Very easy algorithm, Minoux trick still possible, while addresses multiple matroid constraints — but the bound is not that good when there are many matroids.

Matroid Intersection and Bipartite Matching

- Why might we want to do matroid intersection?
- Consider bipartite graph $G = (V, F, E)$. Define two partition matroids $M_V = (E, \mathcal{I}_V)$, and $M_F = (E, \mathcal{I}_F)$.
- Independence in each matroid corresponds to:
  1. $I \in \mathcal{I}_V$ if $|I \cap (V, f)| \leq 1$ for all $f \in F$,
  2. and $I \in \mathcal{I}_F$ if $|I \cap (v, F)| \leq 1$ for all $v \in V$.





- Therefore, a matching in G is simultaneously independent in both M_V and M_F and finding the maximum matching is finding the maximum cardinality set independent in both matroids.
- In bipartite graph case, therefore, can be solved in polynomial time.

Matroid Intersection and Network Communication

- Let $G_1 = (V_1, E)$ and $G_2 = (V_2, E)$ be two graphs on an isomorphic set of edges (lets just give them same names E).
- Consider two cycle matroids associated with these graphs $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$. They might be very different (e.g., an edge might be between two distinct nodes in G_1 but the same edge is a loop in multi-graph G_2.)
- We may wish to find the maximum size edge-induced subgraph that is still forest in both graphs (i.e., adding any edges will create a circuit in either M_1, M_2, or both).
- This is again a matroid intersection problem.

Matroid Intersection and TSP

- Definition: a Hamiltonian cycle is a cycle that passes through each node exactly once.
- Given directed graph G, goal is to find such a Hamiltonian cycle.
- From G with n nodes, create G' with $n+1$ nodes by duplicating (w.l.o.g.) a particular node $v_1 \in V(G)$ to v_1^+, v_1^-, and have all outgoing edges from v_1 come instead from v_1^- and all edges incoming to v_1 go instead to v_1^+.
- Let M_1 be the cycle matroid on G'.
- Let M_2 be the partition matroid having as independent sets those that have no more than one edge leaving any node — i.e., $I \in \mathcal{I}(M_2)$ if $|I \cap \delta^-(v)| \leq 1$ for all $v \in V(G')$.
- Let M_3 be the partition matroid having as independent sets those that have no more than one edge entering any node — i.e., $I \in \mathcal{I}(M_3)$ if $|I \cap \delta^+(v)| \leq 1$ for all $v \in V(G')$.
- Then a Hamiltonian cycle exists iff there is an n-element intersection of M_1, M_2, and M_3.

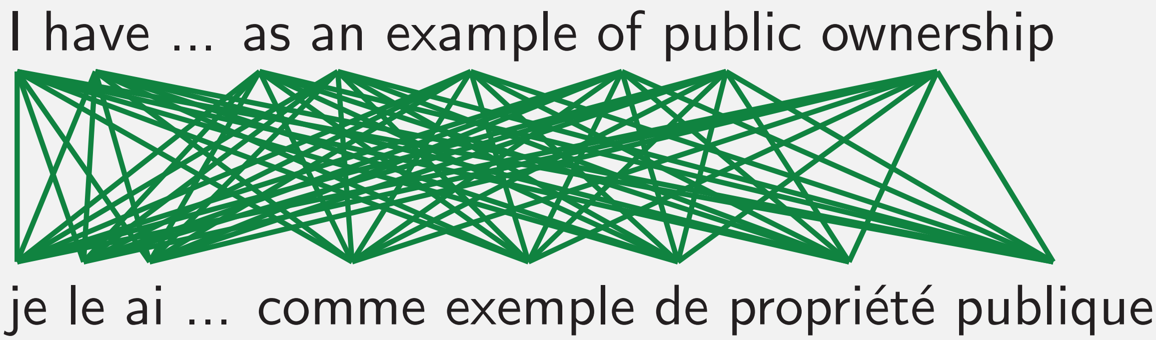
- Recall, the traveling salesperson problem (TSP) is the problem to, given a directed graph, start at a node, visit all cities, and return to the starting point. Optimization version does this tour at minimum cost.
- Since TSP is NP-complete, we obviously can't solve matroid

Greedy over multiple matroids: Generalized Bipartite Matching

- Generalized bipartite matching (i.e., max bipartite matching with submodular costs on the edges). Use two partition matroids (as mentioned earlier in class)
- Useful in natural language processing: Ex. Computer language translation, find an alignment between two language strings.
- Consider bipartite graph $G = (E, F, V)$ where E and F are the left/right set of nodes, respectively, and V is the set of edges.
- E corresponds to, say, an English language sentence and F corresponds to a French language sentence — goal is to form a matching (an alignment) between the two.

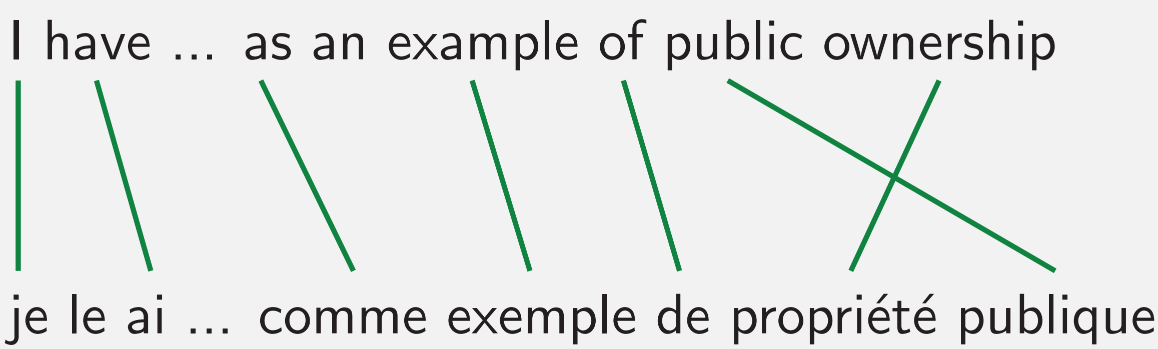
Greedy over > 1 matroids: Multiple Language Alignment

- Consider English string and French string, set up as a bipartite graph.



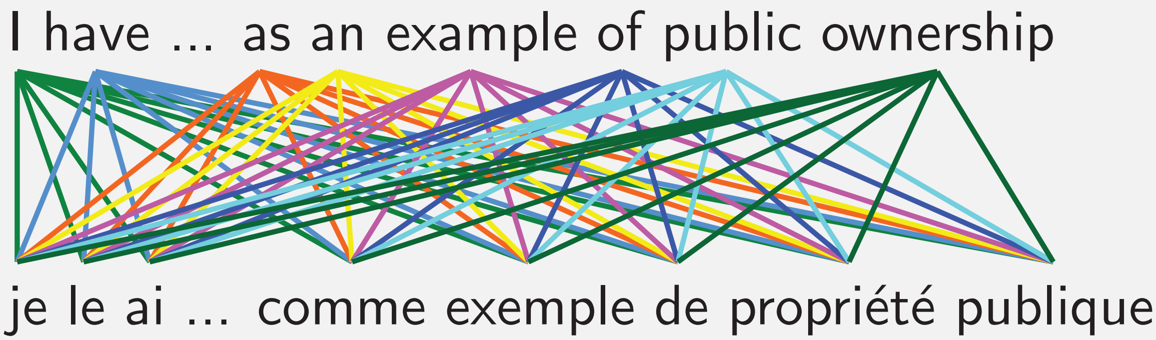
Greedy over > 1 matroids: Multiple Language Alignment

- One possible alignment, a matching, with score as sum of edge weights.



Greedy over > 1 matroids: Multiple Language Alignment

- Edges incident to English words constitute an edge partition

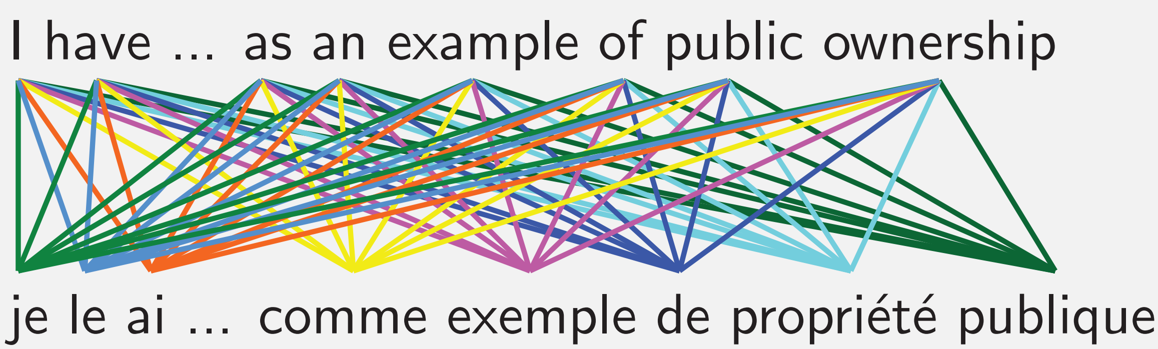


- The two edge partitions can be used to set up two 1-partition matroids on the edges.
- For each matroid, a set of edges is independent only if the set intersects each partition block no more than one time.

Polymatroids, Greedy, and Cardinality Constrained Maximization Curvature Submodular Max w. Other Constraints Submodular Max w. Other Constraints

Greedy over > 1 matroids: Multiple Language Alignment

- Edges incident to French words constitute an edge partition



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Greedy over > 1 matroids: Multiple Language Alignment

- Typical to use bipartite matching to find an alignment between the two language strings.
- As we saw, this is equivalent to two 1-partition matroids and a non-negative modular cost function on the edges.
- We can generalize this using a polymatroid cost function on the edges, and two k-partition matroids, allowing for "fertility" in the models:

Fertility at most 1

Greedy over > 1 matroids: Multiple Language Alignment

- Typical to use bipartite matching to find an alignment between the two language strings.
- As we saw, this is equivalent to two 1-partition matroids and a non-negative modular cost function on the edges.
- We can generalize this using a polymatroid cost function on the edges, and two k-partition matroids, allowing for "fertility" in the models:

Fertility at most 2

Greedy over > 1 matroids: Multiple Language Alignment

- Generalizing further, each block of edges in each partition matroid can have its own "fertility" limit: $\mathcal{I} = \{X \subseteq V : |X \cap V_i| \leq k_i \text{ for all } i = 1, \ldots, \ell\}$.
- Maximizing submodular function subject to multiple matroid constraints addresses this problem.

Greedy over multiple matroids: Submodular Welfare

- Submodular Welfare Maximization: Consider E a set of m goods to be distributed/partitioned among n people ("players").
- Each players has a submodular "valuation" function, $g_i : 2^E \to \mathbb{R}_+$ that measures how "desirable" or "valuable" a given subset $A \subseteq E$ of goods are to that player.
- Assumption: No good can be shared between multiple players, each good must be allocated to a single player.
- Goal of submodular welfare: Partition the goods $E = E_1 \cup E_2 \cup \cdots \cup E_n$ into n blocks in order to maximize the submodular social welfare, measured as:

$$\text{submodular-social-welfare}(E_1, E_2, \ldots, E_n) = \sum_{i=1}^{n} g_i(E_i). \tag{13.28}$$

- We can solve this via submodular maximization subject to multiple matroid independence constraints as we next describe ...

Submodular Welfare: Submodular Max over matroid partition

- Create new ground set E' as disjoint union of n copies of the ground set. I.e.,

$$E' = \underbrace{E \uplus E \uplus \cdots \uplus E}_{n \times} \tag{13.29}$$

- Let $E^{(i)} \subset E'$ be the i^{th} block of E'.
- For any $e \in E$, the corresponding element in $E^{(i)}$ is called $(e, i) \in E^{(i)}$ (each original element is tagged by integer).
- For $e \in E$, define $E_e = \{(e', i) \in E' : e' = e\}$.
- Hence, $\{E_e\}_{e \in E}$ is a partition of E', each block of the partition for one of the original elements in E.
- Create a 1-partition matroid $\mathcal{M} = (E', \mathcal{I})$ where

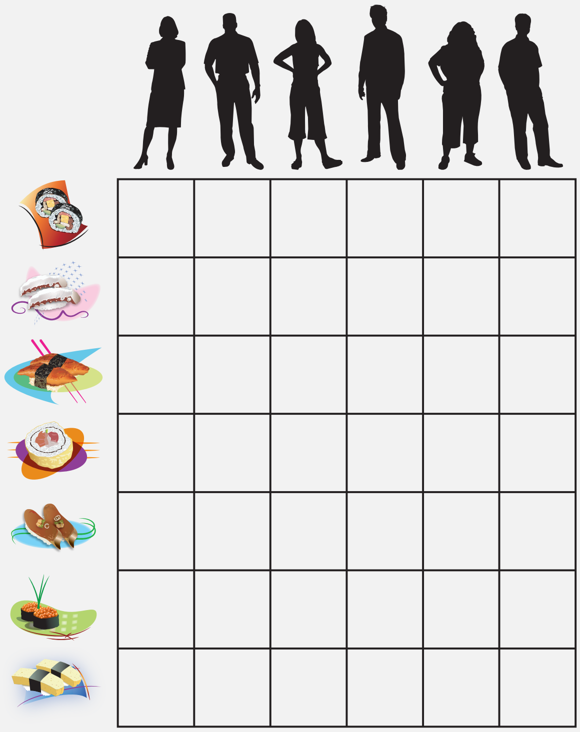
$$\mathcal{I} = \left\{ S \subseteq E' : \forall e \in E, |S \cap E_e| \leq 1 \right\} \tag{13.30}$$

Polymatroids, Greedy, and Cardinality Constrained Maximization | Curvature | Submodular Max w. Other Constraints | Submodular Max w. Other Constraints

Submodular Welfare: Submodular Max over matroid partition

- Hence, S is independent in matroid $\mathcal{M} = (E', I)$ if S uses each original element no more than once.
- Create submodular function $f' : 2^{E'} \to \mathbb{R}_+$ with $f'(S) = \sum_{i=1}^{n} g_i(S \cap E^{(i)})$.
- Submodular welfare maximization becomes matroid constrained submodular max $\max \{f'(S) : S \in \mathcal{I}\}$, so greedy algorithm gives a $1/2$ approximation.

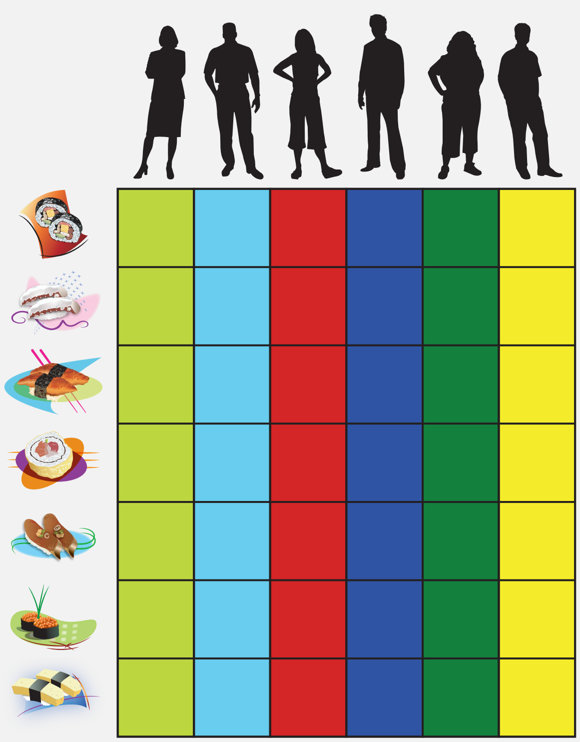
Submodular Social Welfare



- Have $n = 6$ people (who don't like to share) and $|E| = m = 7$ pieces of sushi. E.g., $e \in E$ might be $e =$ "salmon roll".
- Goal: distribute sushi to people to maximize social welfare.
- Ground set disjoint union $E \uplus E \uplus E \uplus E \uplus E \uplus E$.
- Partition matroid partitions: $E_{e_1} \cup E_{e_2} \cup E_{e_3} \cup E_{e_4} \cup E_{e_5} \cup E_{e_6} \cup E_{e_7}$.
- independent allocation
- non-independent allocation

Polymatroids, Greedy, and Cardinality Constrained Maximization | Curvature | Submodular Max w. Other Constraints | Submodular Max w. Other Constraints

Submodular Social Welfare



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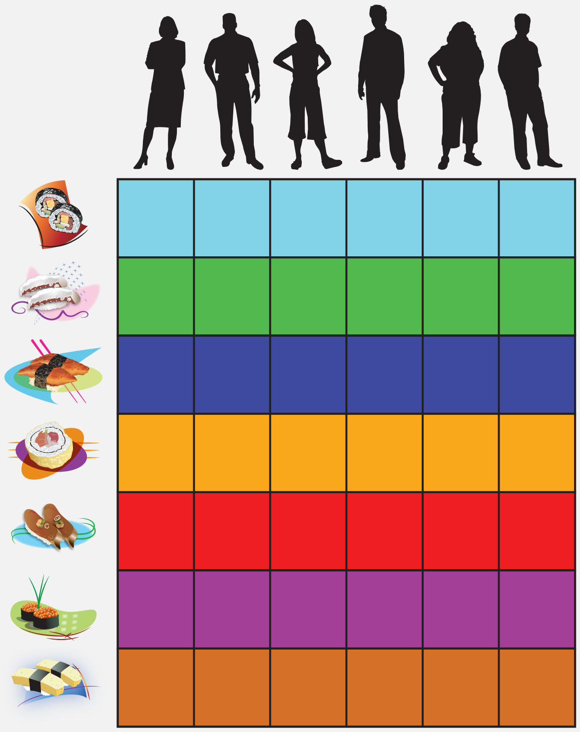

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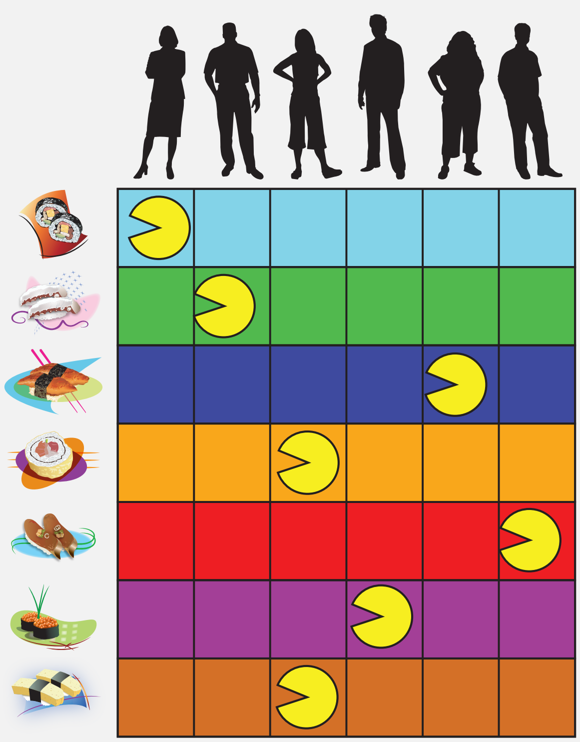

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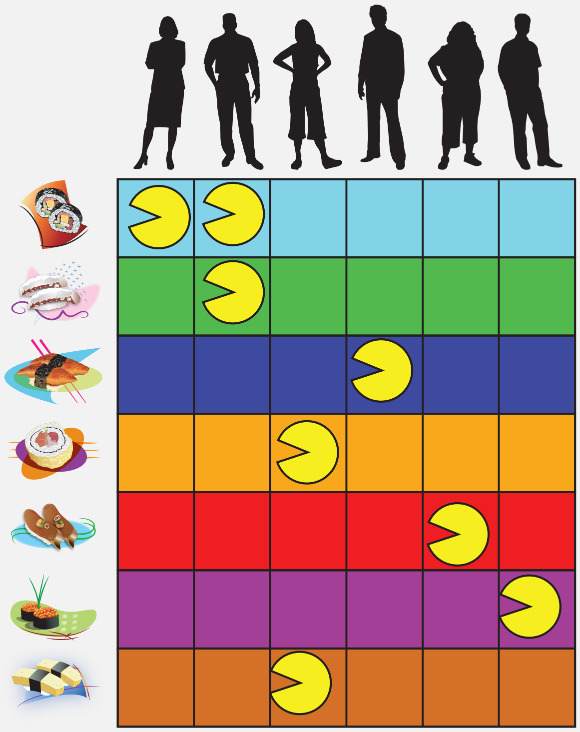

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- independent allocation
- non-independent allocation

Monotone Submodular over Knapsack Constraint

- The constraint $|A| \leq k$ is a simple cardinality constraint.
- Consider a non-negative integral modular function $c : E \to \mathbb{Z}_+$.
- A knapsack constraint would be of the form $c(A) \leq b$ where B is some integer budget that must not be exceeded. That is $\max \{f(A) : A \subseteq V, c(A) \leq b\}$.
- Important: A knapsack constraint yields an independence system (down closed) but it is not a matroid!
- $c(e)$ may be seen as the cost of item e and if $c(e) = 1$ for all e, then we recover the cardinality constraint we saw earlier.

Monotone Submodular over Knapsack Constraint

- Greedy can be seen as choosing the best gain: Starting with $S_0 = \emptyset$, we repeat the following greedy step

$$S_{i+1} = S_i \cup \left\{ \underset{v \in V \setminus S_i}{\operatorname{argmax}} \Big(f(S_i \cup \{v\}) - f(S_i) \Big) \right\} \tag{13.31}$$

  the gain is $f(\{v\}|S_i) = f(S_i + v) - f(S_i)$, so greedy just chooses next the currently unselected element with greatest gain.
- Core idea in knapsack case: Greedy can be extended to choose next whatever looks cost-normalized best, i.e., Starting some initial set S_0, we repeat the following cost-normalized greedy step

$$S_{i+1} = S_i \cup \left\{ \underset{v \in V \setminus S_i}{\operatorname{argmax}} \frac{f(S_i \cup \{v\}) - f(S_i)}{c(v)} \right\} \tag{13.32}$$

  which we repeat until $c(S_{i+1}) > b$ and then take S_i as the solution.

A Knapsack Constraint

- There are a number of ways of getting approximation bounds using this strategy.
- If we run the normalized greedy procedure starting with $S_0 = \emptyset$, and compare the solution found with the max of the singletons $\max_{v \in V} f(\{v\})$, choosing the max, then we get a $(1 - e^{-1/2}) \approx 0.39$ approximation, in $O(n^2)$ time (Minoux trick also possible for further speed)
- Partial enumeration: On the other hand, we can get a $(1 - e^{-1}) \approx 0.63$ approximation in $O(n^5)$ time if we run the above procedure starting from all sets of cardinality three (so restart for all S_0 such that $|S_0| = 3$), and compare that with the best singleton and pairwise solution.
- Extending something similar to this to d simultaneous knapsack constraints is possible as well.

Local Search Algorithms

From J. Vondrak

- Local search involves switching up to t elements, as long as it provides a (non-trivial) improvement; can iterate in several phases. Some examples follow:
- 1/3 approximation to unconstrained non-monotone maximization [Feige, Mirrokni, Vondrak, 2007]
- $1/(k + 2 + \frac{1}{k} + \delta_t)$ approximation for non-monotone maximization subject to k matroids [Lee, Mirrokni, Nagarajan, Sviridenko, 2009]
- $1/(k + \delta_t)$ approximation for monotone submodular maximization subject to $k \geq 2$ matroids [Lee, Sviridenko, Vondrak, 2010].

What About Non-monotone

- Alternatively, we may wish to maximize non-monotone submodular functions. This includes of course graph cuts, and this problem is APX-hard, so maximizing non-monotone functions, even unconstrainedly, is hard.
- If f is an arbitrary submodular function (so neither polymatroidal, nor necessarily positive or negative), then verifying if the maximum of f is positive or negative is already NP-hard.
- Therefore, submodular function max in such case is inapproximable unless P=NP (since any such procedure would give us the sign of the max).
- Thus, any approximation algorithm must be for unipolar submodular functions. E.g., non-negative but otherwise arbitrary submodular functions.
- We may get a $(\frac{1}{3} - \frac{\epsilon}{n})$ approximation for maximizing non-monotone non-negative submodular functions, with most $O(\frac{1}{\epsilon}n^3 \log n)$ function calls using approximate local maxima.

Submodularity and local optima

- Given any submodular function f, a set $S \subseteq V$ is a local maximum of f if $f(S - v) \leq f(S)$ for all $v \in S$ and $f(S + v) \leq f(S)$ for all $v \in V \setminus S$ (i.e., local in a Hamming ball of radius 1).
- The following interesting result is true for any submodular function:

Lemma 13.6.1

Given a submodular function f, if S is a local maximum of f, and $I \subseteq S$ or $I \supseteq S$, then $f(I) \leq f(S)$.

- Idea of proof: Given $v_1, v_2 \in S$, suppose $f(S - v_1) \leq f(S)$ and $f(S - v_2) \leq f(S)$. Submodularity requires $f(S - v_1) + f(S - v_2) \geq f(S) + f(S - v_1 - v_2)$ which would be impossible unless $f(S - v_1 - v_2) \leq f(S)$.
- Similarly, given $v_1, v_2 \notin S$, and $f(S + v_1) \leq f(S)$ and $f(S + v_2) \leq f(S)$. Submodularity requires $f(S + v_1) + f(S + v_2) \geq f(S) + f(S + v_1 + v_2)$ which requires $f(S + v_1 + v_2) \leq f(S)$.

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Given a submodular function f, if S is a local maximum of f, and $I \subseteq S$ or $I \supseteq S$, then $f(I) \leq f(S)$.

- In other words, once we have identified a local maximum, the two intervals in the Boolean lattice $[\emptyset, S]$ and $[S, V]$ can be ruled out as a possible improvement over S.
- Finding a local maximum is already hard (PLS-complete), but it is possible to find an approximate local maximum relatively efficiently.
- This is the approach that yields the $(\frac{1}{3} - \frac{\epsilon}{n})$ approximation algorithm.

Linear time algorithm unconstrained non-monotone max

- Tight randomized tight $1/2$ approximation algorithm for unconstrained non-monotone non-negative submodular maximization.
- Buchbinder, Feldman, Naor, Schwartz 2012. Recall $[a]_+ = \max(a, 0)$.

Algorithm 3: Randomized Linear-time non-monotone submodular max

```
Set L ← ∅ ; U ← V      /* Lower L, upper U. Invariant: L ⊆ U */ ;
Order elements of V = (v_1, v_2, ..., v_n) arbitrarily ;
for i ← 0 ... |V| do
    a ← [f(v_i|L)]_+; b ← [−f(U|U \ {v_i})]_+ ;
    if a = b = 0 then p ← 1/2 ;
    ;
    else p ← a/(a + b);
    ;
    if Flip of coin with Pr(heads) = p draws heads then
        L ← L ∪ {v_i} ;
    Otherwise /* if the coin drew tails, an event with prob. 1 − p */
        U ← U \ {v}
return L (which is the same as U at this point)
```


Linear time algorithm unconstrained non-monotone max

- Each "sweep" of the algorithm is $O(n)$.
- Running the algorithm $1\times$ (with an arbitrary variable order) results in a $1/3$ approximation.
- The $1/2$ guarantee is in expected value (the expected solution has the $1/2$ guarantee).
- In practice, run it multiple times, each with a different random permutation of the elements, and then take the cumulative best.
- It may be possible to choose the random order smartly to get better results in practice.

More general still: multiple constraints different types

- In the past several years, there has been a plethora of papers on maximizing both monotone and non-monotone submodular functions under various combinations of one or more knapsack and/or matroid constraints.
- The approximation quality is usually some function of the number of matroids, and is often not a function of the number of knapsacks.
- Often the computational costs of the algorithms are prohibitive (e.g., exponential in k) with large constants, so these algorithms might not scale.
- On the other hand, these algorithms offer deep and interesting intuition into submodular functions, beyond what we have covered here.

Some results on submodular maximization

- As we've seen, we can get $1 - 1/e$ for non-negative monotone submodular (polymatroid) functions with greedy algorithm under cardinality constraints, and this is tight.
- For general matroid, greedy reduces to $1/2$ approximation (as we've seen).
- We can recover $1 - 1/e$ approximation using the continuous greedy algorithm on the multilinear extension and then using pipage rounding to re-integerize the solution (see J. Vondrak's publications).
- More general constraints are possible too, as we see on the next table (for references, see Jan Vondrak's publications `http://theory.stanford.edu/~jvondrak/`).

Submodular Max Summary - 2012: From J. Vondrak

Monotone Maximization

Constraint	Approximation	Hardness	Technique
$\|S\| \leq k$	$1-1/e$	$1-1/e$	greedy
matroid	$1-1/e$	$1-1/e$	multilinear ext.
$O(1)$ knapsacks	$1-1/e$	$1-1/e$	multilinear ext.
k matroids	$k+\epsilon$	$k/\log k$	local search
k matroids and $O(1)$ knapsacks	$O(k)$	$k/\log k$	multilinear ext.

Nonmonotone Maximization

Constraint	Approximation	Hardness	Technique
Unconstrained	$1/2$	$1/2$	combinatorial
matroid	$1/e$	0.48	multilinear ext.
$O(1)$ knapsacks	$1/e$	0.49	multilinear ext.
k matroids	$k+O(1)$	$k/\log k$	local search
k matroids and $O(1)$ knapsacks	$O(k)$	$k/\log k$	multilinear ext.