

# Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture 19 —

[http://www.ee.washington.edu/people/faculty/bilmes/classes/ee563\\_spring\\_2018/](http://www.ee.washington.edu/people/faculty/bilmes/classes/ee563_spring_2018/)

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June 6th, 2018



$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

$$-f(A) + 2f(C) + f(B) \quad -f(A) + f(C) + f(B) \quad -f(A \cap B)$$



# Announcements, Assignments, and Reminders

- Take home final exam (like long homework). Due Friday, June 8th, 4:00pm via our assignment dropbox (<https://canvas.uw.edu/courses/1216339/assignments>).
- Get started now. At least read through everything and ask any questions you might have.
- As always, if you have any questions about anything, please ask them via our discussion board ([https://canvas.uw.edu/courses/1216339/discussion\\_topics](https://canvas.uw.edu/courses/1216339/discussion_topics)). Can meet at odd hours via zoom (send message on canvas to schedule time to chat).

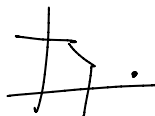
# Class Road Map - EE563

- L1(3/26): Motivation, Applications, & Basic Definitions,
- L2(3/28): Machine Learning Apps (diversity, complexity, parameter, learning target, surrogate).
- L3(4/2): Info theory exs, more apps, definitions, graph/combinatorial examples
- L4(4/4): Graph and Combinatorial Examples, Matrix Rank, Examples and Properties, visualizations
- L5(4/9): More Examples/Properties/ Other Submodular Defs., Independence,
- L6(4/11): Matroids, Matroid Examples, Matroid Rank, Partition/Laminar Matroids
- L7(4/16): Laminar Matroids, System of Distinct Reps, Transversals, Transversal Matroid, Matroid Representation, Dual Matroids
- L8(4/18): Dual Matroids, Other Matroid Properties, Combinatorial Geometries, Matroids and Greedy.
- L9(4/23): Polyhedra, Matroid Polytopes, Matroids  $\rightarrow$  Polymatroids
- L10(4/29): Matroids  $\rightarrow$  Polymatroids, Polymatroids, Polymatroids and Greedy,
- L11(4/30): Polymatroids, Polymatroids and Greedy
- L12(5/2): Polymatroids and Greedy, Extreme Points, Cardinality Constrained Maximization
- L13(5/7): Constrained Submodular Maximization
- L14(5/9): Submodular Max w. Other Constraints, Cont. Extensions, Lovasz Extension
- L15(5/14): Cont. Extensions, Lovasz Extension, Choquet Integration, Properties
- L16(5/16): More Lovasz extension, Choquet, defs/props, examples, multilinear extension
- L17(5/21): Finish L.E., Multilinear Extension, Submodular Max/polyhedral approaches, Most Violated inequality, Still More on Matroids, Closure/Sat
- L-(5/28): Memorial Day (holiday)
- L18(5/30): Closure/Sat, Fund. Circuit/Dep
- L19(6/6): Fund. Circuit/Dep, Min-Norm Point Definitions, Review & Support for Min-Norm, Proof that min-norm gives optimal, Computing Min-Norm Vector for  $B_f$  maximization.

Last day of instruction, June 1st. Finals Week: June 2-8, 2018.

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- Most violated inequality  $\max \{x(A) - f(A) : A \subseteq E\}$



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- $e$ -containing tight sets
- dep function & fundamental circuit of a matroid

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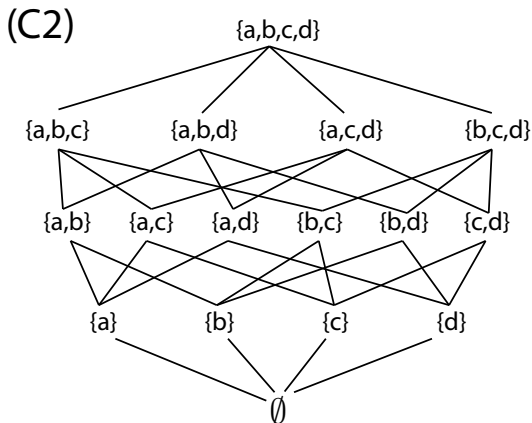
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- Minimal  $e$ -containing  $x$ -tight set/polymatroidal fundamental circuit/  
 For  $x \in P_f$ ,  

$$\text{dep}(x, e) = \begin{cases} \bigcap \{A : e \in A \subseteq E, x(A) = f(A)\} & \text{if } e \in \text{sat}(x) \\ \emptyset & \text{else} \end{cases}$$

$$= \{e' : \exists \alpha > 0, \text{ s.t. } x + \alpha(\mathbf{1}_e - \mathbf{1}_{e'}) \in P_f\}$$

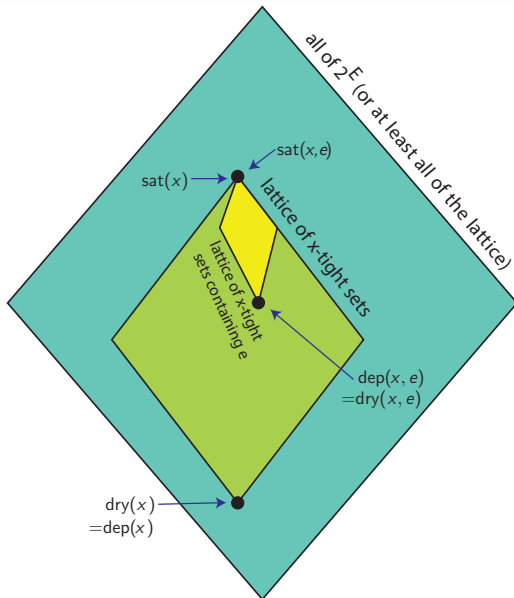
# dep and sat in a lattice

- Given some  $x \in P_f$ ,
- The picture on the right summarizes the relationships between the lattices and sublattices.
- Note,  $\text{dep}(x, e) \supseteq \text{dep}(x) = \bigcap \{A : x(A) = f(A)\}$ .
- In fact,  $\text{sat}(x, e) = \text{sat}(x)$ .  
Why?
- Example lattice on 4 elements.



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# Submodular Function Minimization (SFM) and Min-Norm

- We saw that SFM can be used to solve most violated inequality problems for a given  $x \in P_f$  and, in general, SFM can solve the question “Is  $x \in P_f$ ” by seeing if  $x$  violates any inequality (if the most violated one is negative, solution to SFM, then  $x \in P_f$ ).
- Unconstrained SFM,  $\min_{A \subseteq V} f(A)$  solves many other problems as well in combinatorial optimization, machine learning, and other fields.
- We next study an algorithm, the “Fujishige-Wolfe Algorithm”, or what is known as the “Minimum Norm Point” algorithm, which is an active set method to do this, and one that in practice works about as well as anything else people (so far) have tried for general purpose SFM.
- Note special case SFM can be much faster.

# Min-Norm Point: Definition

- Consider the optimization:

$$\text{minimize} \quad \|x\|_2^2 \quad (19.1a)$$

$$\text{subject to} \quad x \in B_f \quad (19.1b)$$

where  $B_f$  is the base polytope of submodular  $f$ , and  $\|x\|_2^2 = \sum_{e \in E} x(e)^2$  is the squared 2-norm. Let  $x^*$  be the optimal solution.

$$f(A) = \sqrt{|A|} \Rightarrow B_f \text{ has } n! \text{ vertices.}$$

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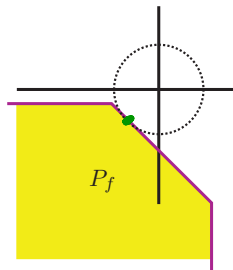
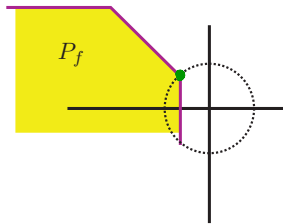
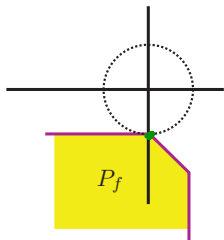
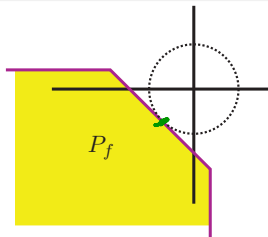
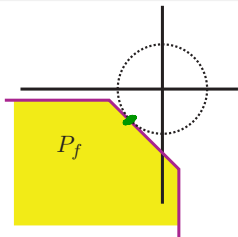
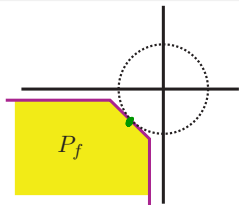
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- Note,  $x^*$  is **the** unique optimal solution since we have a strictly convex objective over a set of convex constraints.
- $x^*$  is called the **minimum norm point** of the base polytope.



# Min-Norm Point: Examples



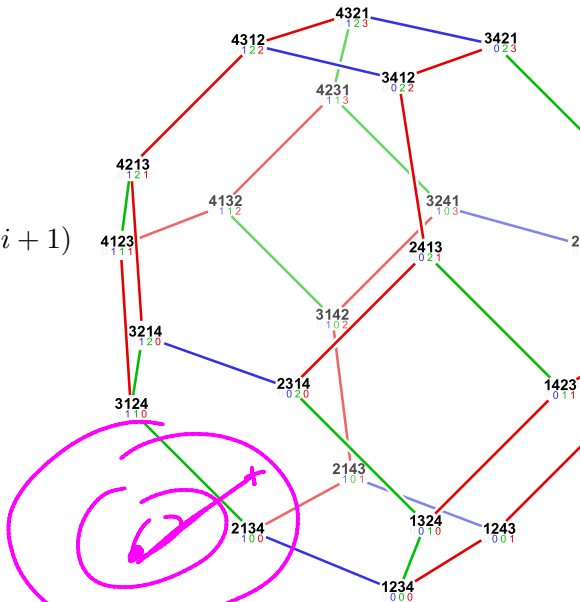
# Ex: 3D base $B_f$ : permutahedron

- Consider submodular function  $f : 2^V \rightarrow \mathbb{R}$  with  $n = |V| = 4$ , and for  $X \subseteq V$ , concave  $g$ ,

$$f(X) = g(|X|) = \sum_{i=1}^{|X|} (n - i + 1)$$

$$= |X| \left( n - \frac{|X| - 1}{2} \right)$$

- Then  $B_f$  is a 3D polytope, and in this particular case gives us a permutahedron with 24 distinct extreme points, on the right (from wikipedia).



# Min-Norm Point and Submodular Function Minimization

- Given optimal solution  $x^*$  to  $[\min \|x\|_2^2 \text{ s.t. } x \in B_f]$ , and consider:

$$y^* = x^* \wedge 0 = (\min(x^*(e), 0) | e \in E), \quad (19.2)$$

$$A_- = \{e : x^*(e) < 0\}, \quad (19.3)$$

$$A_0 = \{e : x^*(e) \leq 0\}. \quad (19.4)$$

$$x^* = [3, 4, 0, -1, -5, 2]$$

$$y^* = [0, 0, 0, -1, -5, 2]$$

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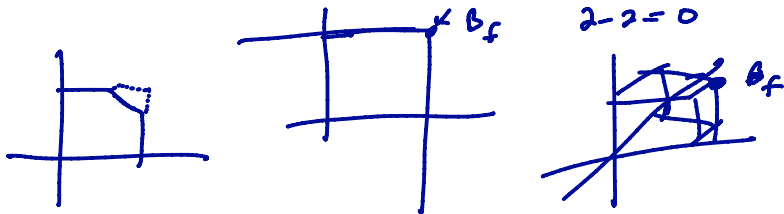
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- It turns out, these quantities will solve the submodular function minimization problem, as we now show.
- The proof is nice since it uses the tools we've been recently developing.

# More about the base $B_f$

## Theorem 19.5.1

Let  $f$  be a polymatroid function and suppose that  $E$  can be partitioned into  $(E_1, E_2, \dots, E_k)$  such that  $f(A) = \sum_{i=1}^k f(A \cap E_i)$  for all  $A \subseteq E$ , and  $k$  is maximum. Then the base polytope  $B_f = \{x \in P_f : x(E) = f(E)\}$  (the  $E$ -tight subset of  $P_f$ ) has dimension  $|E| - k$ .



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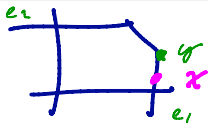
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- In fact, every  $x \in P_f$  is dominated by  $x \leq y \in B_f$ .

## Theorem 19.5.2

If  $x \in P_f$  and  $T$  is tight for  $x$  (meaning  $x(T) = f(T)$ ), then there exists  $y \in B_f$  with  $x \leq y$  and  $y(e) = x(e)$  for  $e \in T$ .



$$x(T) = f(T) = f(e_1, 3) \quad T = \{e_1, 3\}$$

$$x(e_2) < y(e_2)$$



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- We leave the proof as an exercise.

# Review from Lecture 12

The following slide repeats Theorem 12.3.2 from lecture 12 and is one of the most important theorems in submodular theory.

# A polymatroid function's polyhedron is a polymatroid.

## Theorem 19.5.1

Let  $f$  be a submodular function defined on subsets of  $E$ . For any  $x \in \mathbb{R}^E$ , we have:

$$\text{rank}(x) = \max (y(E) : y \leq x, y \in P_f) = \min (x(A) + f(E \setminus A) : A \subseteq E) \quad (19.1)$$

Essentially the same theorem as Theorem 10.4.1, but note  $P_f$  rather than  $P_f^+$ . Taking  $x = 0$  we get:



## Corollary 19.5.2

Let  $f$  be a submodular function defined on subsets of  $E$ . We have:

$$\text{rank}(0) = \max (y(E) : y \leq 0, y \in P_f) = \min (f(A) : A \subseteq E) \quad (19.2)$$

# Modified max-min theorem

- Min-max theorem (Thm 12.3.2) restated for  $x = 0$ .

$$\max \{y(E) \mid y \in P_f, y \leq 0\} = \min \{f(X) \mid X \subseteq V\} \quad (19.7)$$

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## Theorem 19.5.3 (Edmonds-1970)

$$\min \{f(X) \mid X \subseteq E\} = \max \{x^-(E) \mid x \in B_f\} \quad (19.8)$$

where  $x^-(e) = \min \{x(e), 0\}$  for  $e \in E$ .

# Modified max-min theorem

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## Proof via the Lovász ext.

$$\min \{f(X) \mid X \subseteq E\} = \min_{w \in [0,1]^E} \tilde{f}(w) = \min_{w \in [0,1]^E} \max_{x \in P_f} w^\top x \quad (19.9)$$

$$= \min_{w \in [0,1]^E} \max_{x \in B_f} w^\top x \quad (19.10)$$

$$= \max_{x \in B_f} \min_{w \in [0,1]^E} w^\top x \quad (19.11)$$

$$= \max_{x \in B_f} x^-(E) \quad (19.12)$$



# Alternate proof of modified max-min theorem

We start directly from Theorem 12.3.2.

$$\max (y(E) : y \leq 0, y \in P_f) = \min (f(A) : A \subseteq E) \quad (19.16)$$

Given  $y \in \mathbb{R}^E$ , define  $y^- \in \mathbb{R}^E$  with  $y^-(e) = \min \{y(e), 0\}$  for  $e \in E$ .

$$\max (y(E) : y \leq 0, y \in P_f) = \max (y^-(E) : y \leq 0, y \in P_f) \quad (19.17)$$

$$= \max (y^-(E) : y \in P_f) \quad (19.18)$$

$$= \max (y^-(E) : y \in B_f) \quad (19.19)$$

The first equality follows since  $y \leq 0$ . The second equality (together with the first) shown on following slide. The third equality follows since for any  $x \in P_f$  there exists a  $y \in B_f$  with  $x \leq y$  (follows from Theorem 19.5.2).

# Alternate proof of modified max-min theorem

Consider the following two problems:

$$\max \sum_{e \in E} y(e) \quad (19.20a)$$

$$\text{s.t. } y \leq x \quad (19.20b)$$

$$y \in P \quad (19.20c)$$

$$\max \left( \sum_{e \in E} \min(y(e), x(e)) \right) \quad (19.21a)$$

$$\text{s.t. } y \in P \quad (19.21b)$$

- Solutions identical cost. Let  $y_1^*$  be l.h.s. OPT and  $y_2^*$  be r.h.s. OPT.
- Consider  $y_1^*$  as r.h.s. solution and suppose it is worse than r.h.s. OPT:

$$\sum_{e \in E} \min(y_1^*(e), x(e)) < \sum_{e \in E} \min(y_2^*(e), x(e)) \quad (19.22)$$



# Alternate proof of modified max-min theorem

Consider the following two problems:

$$\max \sum_{e \in E} y(e) \quad (19.20a)$$

$$\text{s.t. } y \leq x \quad (19.20b)$$

$$y \in P \quad (19.20c)$$

$$\max \sum_{e \in E} \min(y(e), x(e)) \quad (19.21a)$$

$$\text{s.t. } y \in P \quad (19.21b)$$

- Solutions identical cost. Let  $y_1^*$  be l.h.s. OPT and  $y_2^*$  be r.h.s. OPT.
- Consider  $y_1^*$  as r.h.s. solution and suppose it is worse than r.h.s. OPT:

$$\sum_{e \in E} \min(y_1^*(e), x(e)) < \sum_{e \in E} \min(y_2^*(e), x(e)) \quad (19.22)$$

- Hence,  $\exists e'$  s.t.  $y_1^*(e') < \min(y_2^*(e'), x(e'))$ . Recall  $y_1^*, y_2^* \in P$ .

# Alternate proof of modified max-min theorem

Consider the following two problems:

$$\max \sum_{e \in E} y(e) \quad (19.20a)$$

$$\text{s.t. } y \leq x \quad (19.20b)$$

$$y \in P \quad (19.20c)$$

$$\max \left[ \sum_{e \in E} \min(y(e), x(e)) \right] \quad (19.21a)$$

$$\text{s.t. } y \in P \quad (19.21b)$$

- Solutions identical cost. Let  $y_1^*$  be l.h.s. OPT and  $y_2^*$  be r.h.s. OPT.
- Consider  $y_1^*$  as r.h.s. solution and suppose it is worse than r.h.s. OPT:

$$\sum_{e \in E} \min(y_1^*(e), x(e)) < \sum_{e \in E} \min(y_2^*(e), x(e)) \quad (19.22)$$

- Hence,  $\exists e'$  s.t.  $y_1^*(e') < \min(y_2^*(e'), x(e'))$ . Recall  $y_1^*, y_2^* \in P$ .
- This implies  $\sum_{e \neq e'} y_1^*(e) + y_1^*(e') < \sum_{e \neq e'} y_1^*(e) + \min(y_2^*(e'), x(e'))$ , better feasible solution to l.h.s., contradicting  $y_1^*$ 's optimality for l.h.s.

$$y_1^*(e') \rightarrow \min(y_2^*(e'), x(e')) = \bar{y}_1^*(e')$$

# Alternate proof of modified max-min theorem

Consider the following two problems:

$$\max \sum_{e \in E} y(e) \quad (19.20a)$$

$$\text{s.t. } y \leq x \quad (19.20b)$$

$$y \in P \quad (19.20c)$$

$$\max \sum_{e \in E} \min(y(e), x(e)) \quad (19.21a)$$

$$\text{s.t. } y \in P \quad (19.21b)$$

- Solutions identical cost. Let  $y_1^*$  be l.h.s. OPT and  $y_2^*$  be r.h.s. OPT.
- Similarly, consider  $y_2^*$  as l.h.s. solution, suppose worse than l.h.s. OPT

$$\sum_{e \in E} y_2^*(e) < \sum_{e \in E} y_1^*(e) \quad (19.22)$$

# Alternate proof of modified max-min theorem

Consider the following two problems:

$$\max \sum_{e \in E} y(e) \quad (19.20a)$$

$$\text{s.t. } y \leq x \quad (19.20b)$$

$$y \in P \quad (19.20c)$$

$$\max \sum_{e \in E} \min(y(e), x(e)) \quad (19.21a)$$

$$\text{s.t. } y \in P \quad (19.21b)$$

- Solutions identical cost. Let  $y_1^*$  be l.h.s. OPT and  $y_2^*$  be r.h.s. OPT.
- Similarly, consider  $y_2^*$  as l.h.s. solution, suppose worse than l.h.s. OPT

$$\sum_{e \in E} y_2^*(e) < \sum_{e \in E} y_1^*(e) \quad (19.22)$$

- Then  $\exists e'$  such that  $y_2^*(e') < y_1^*(e') \leq x(e')$ .

# Alternate proof of modified max-min theorem

Consider the following two problems:

$$\max \sum_{e \in E} y(e) \quad (19.20a)$$

$$\text{s.t. } y \leq x \quad (19.20b)$$

$$y \in P \quad (19.20c)$$

$$\max \sum_{e \in E} \min(y(e), x(e)) \quad (19.21a)$$

$$\text{s.t. } y \in P \quad (19.21b)$$

- Solutions identical cost. Let  $y_1^*$  be l.h.s. OPT and  $y_2^*$  be r.h.s. OPT.
- Similarly, consider  $y_2^*$  as l.h.s. solution, suppose worse than l.h.s. OPT

$$\sum_{e \in E} y_2^*(e) < \sum_{e \in E} y_1^*(e) \quad (19.22)$$

- Then  $\exists e'$  such that  $y_2^*(e') < y_1^*(e') \leq x(e')$ .
- This implies that replacing  $y_2^*(e')$ 's value with  $y_1^*(e')$  is still feasible for r.h.s. but better, contradicting  $y_2^*$ 's optimality.

# Alternate proof of modified max-min theorem

Consider the following two problems:

$$\max \sum_{e \in E} y(e) \quad (19.20a)$$

$$\text{s.t. } y \leq x \quad (19.20b)$$

$$y \in P \quad (19.20c)$$

$$\max \sum_{e \in E} \min(y(e), x(e)) \quad (19.21a)$$

$$\text{s.t. } y \in P \quad (19.21b)$$

- Solutions identical cost. Let  $y_1^*$  be l.h.s. OPT and  $y_2^*$  be r.h.s. OPT.
- Similarly, consider  $y_2^*$  as l.h.s. solution, suppose worse than l.h.s. OPT

$$\sum_{e \in E} y_2^*(e) < \sum_{e \in E} y_1^*(e) \quad (19.22)$$

- Then  $\exists e'$  such that  $y_2^*(e') < y_1^*(e') \leq x(e')$ .
- This implies that replacing  $y_2^*(e')$ 's value with  $y_1^*(e')$  is still feasible for r.h.s. but better, contradicting  $y_2^*$ 's optimality.
- Hence, from previous slide, taking  $x = 0$ ,  $\max(y(E) : y \leq 0, y \in P_f) = \max(y^-(E) : y \in P_f) = \max(y^-(E) : y \in B_f)$

$$\min \{w^\top x : x \in B_f\}$$

- Recall that the greedy algorithm solves, for  $w \in \mathbb{R}_+^E$

$$\max \{w^\top x | x \in P_f\} = \max \{w^\top x | x \in B_f\} \quad (19.23)$$

since for all  $x \in P_f$ , there exists  $y \geq x$  with  $y \in B_f$ .

$$\min \{w^\top x : x \in B_f\}$$

- Recall that the greedy algorithm solves, for  $w \in \mathbb{R}_+^E$

$$\max \{w^\top x | x \in P_f\} = \max \{w^\top x | x \in B_f\} \quad (19.23)$$

since for all  $x \in P_f$ , there exists  $y \geq x$  with  $y \in B_f$ .

- For arbitrary  $w \in \mathbb{R}^E$ , greedy algorithm will also solve:

$$\max \{w^\top x | x \in B_f\} \quad (19.24)$$



$$\min \{w^\top x : x \in B_f\}$$

- Recall that the greedy algorithm solves, for  $w \in \mathbb{R}_+^E$

$$\max \{w^\top x | x \in P_f\} = \max \{w^\top x | x \in B_f\} \quad (19.23)$$

since for all  $x \in P_f$ , there exists  $y \geq x$  with  $y \in B_f$ .

- For arbitrary  $w \in \mathbb{R}^E$ , greedy algorithm will also solve:

$$\max \{w^\top x | x \in B_f\}$$

$$\begin{array}{l} \sigma \text{ s.t.} \\ w_{\sigma_1} \geq w_{\sigma_2} \geq \dots \geq w_{\sigma_m} \end{array} \quad (19.24)$$

$$x(\sigma_1) = f(\sigma_1)$$

$$x(\sigma_2) = f(\sigma_2 | \sigma_1)$$

$$x(\sigma_3) = f(\sigma_3 | \sigma_1, \dots, \sigma_{2-1})$$

$$\dots$$

$$x(\sigma_m) = \dots$$

- Also, since  $w \in \mathbb{R}^E$  is arbitrary, and since

$$\min \{w^\top x | x \in B_f\} = - \max \{-w^\top x | x \in B_f\} \quad (19.25)$$

the greedy algorithm using ordering  $(e_1, e_2, \dots, e_m)$  such that

$$w(e_1) \leq w(e_2) \leq \dots \leq w(e_m) \quad (19.26)$$

will solve l.h.s. of Equation (19.25).

Greedy solves  $\max \{w^\top x \mid x \in B_f\}$  for arbitrary  $w \in \mathbb{R}^E$

Let  $f(A)$  be arbitrary submodular function, and  $f(A) = f'(A) - m(A)$  where  $f'$  is polymatroidal, and  $w \in \mathbb{R}^E$ .

$$\begin{aligned}
 \max \{w^\top x \mid x \in B_f\} &= \max \{w^\top x \mid x(A) \leq f(A) \forall A, x(E) = f(E)\} \\
 &= \max \{w^\top x \mid x(A) \leq f'(A) - m(A) \forall A, x(E) = f'(E) - m(E)\} \\
 &= \max \{w^\top x \mid x(A) + m(A) \leq f'(A) \forall A, x(E) + m(E) = f'(E)\} \\
 &= \max \{w^\top x + w^\top m \mid \\
 &\quad x(A) + m(A) \leq f'(A) \forall A, x(E) + m(E) = f'(E)\} - w^\top m \\
 &= \max \{w^\top y \mid y \in B_{f'}\} - w^\top m \\
 &= w^\top y^* - w^\top m = w^\top (y^* - m)
 \end{aligned}$$

$\sigma$   
 $y(v_1) = f'(v_1) = f(v_1) + m(v_1)$   
 $y(v_2) = f'(v_2) = f(v_2) + m(v_2)$

where  $y = x + m$ , so that  $x^* = y^* - m$ .

So  $y^*$  uses greedy algorithm with positive orthant  $B_{f'}$ . To show, we use Theorem 11.4.1 in Lecture 11, but we don't require  $y \geq 0$ , and don't stop when  $w$  goes negative to ensure  $y^* \in B_{f'}$ . Then when we subtract off  $m$  from  $y^*$ , we get solution to the original problem.

# Min-Norm Point and Submodular Function Minimization

- Given optimal solution  $x^*$  to  $[\min \|x\|_2^2 \text{ s.t. } x \in B_f]$ , and consider:

$$y^* = x^* \wedge 0 = (\min(x^*(e), 0) | e \in E), \quad (19.2)$$

$$A_- = \{e : x^*(e) < 0\}, \quad (19.3)$$

$$A_0 = \{e : x^*(e) \leq 0\}. \quad (19.4)$$

- Thus, we immediately have that:

$$A_- \subseteq A_0 \quad (19.5)$$

and that

$$x^*(A_-) = x^*(A_0) = y^*(A_-) = y^*(A_0) \quad (19.6)$$

- It turns out, these quantities will solve the submodular function minimization problem, as we now show.
- The proof is nice since it uses the tools we've been recently developing.

# Min-Norm Point and SFM

## Theorem 19.6.1

Let  $x^*$ ,  $y^*$ ,  $A_-$ , and  $A_0$  be as given. Then  $y^*$  is a maximizer of the l.h.s. of Eqn. (19.7). Moreover,  $A_-$  is the unique minimal minimizer of  $f$  and  $A_0$  is the unique maximal minimizer of  $f$ .

## Proof.

- First note, since  $x^* \in B_f$ , we have  $x^*(E) = f(E)$ , meaning  $\text{sat}(x^*) = E$ . Thus, we may consider any  $e \in E$  within  $\text{dep}(x^*, e)$ .



# Min-Norm Point and SFM

## Theorem 19.6.1

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- First note, since  $x^* \in B_f$ , we have  $x^*(E) = f(E)$ , meaning  $\text{sat}(x^*) = E$ . Thus, we may consider any  $e \in E$  within  $\text{dep}(x^*, e)$ .
- Consider any pair  $(e, e')$  with  $e \in A_-$  and  $e' \in \text{dep}(x^*, e)$ . Then  $x^*(e) < 0$ , and  $\exists \alpha > 0$  s.t.  $x^* + \alpha \mathbf{1}_e - \alpha \mathbf{1}_{e'} \in P_f$ .

# Min-Norm Point and SFM

## Theorem 19.6.1

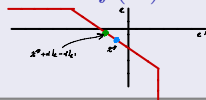
Let  $x^*$ ,  $y^*$ ,  $A_-$ , and  $A_0$  be as given. Then  $y^*$  is a maximizer of the l.h.s. of Eqn. (19.7). Moreover,  $A_-$  is the unique minimal minimizer of  $f$  and  $A_0$  is the unique maximal minimizer of  $f$ .

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- First note, since  $x^* \in B_f$ , we have  $x^*(E) = f(E)$ , meaning  $\text{sat}(x^*) = E$ . Thus, we may consider any  $e \in E$  within  $\text{dep}(x^*, e)$ .
- Consider any pair  $(e, e')$  with  $e \in A_-$  and  $e' \in \text{dep}(x^*, e)$ . Then  $x^*(e) < 0$ , and  $\exists \alpha > 0$  s.t.  $x^* + \alpha \mathbf{1}_e - \alpha \mathbf{1}_{e'} \in P_f$ .
- We have  $x^*(E) = f(E)$  and  $x^*$  is minimum in  $l_2$  sense. We have  $(x^* + \alpha \mathbf{1}_e - \alpha \mathbf{1}_{e'}) \in P_f$ , and in fact

$$(x^* + \alpha \mathbf{1}_e - \alpha \mathbf{1}_{e'})(E) = x^*(E) + \alpha - \alpha = f(E) \quad (19.27)$$

so  $x^* + \alpha \mathbf{1}_e - \alpha \mathbf{1}_{e'} \in B_f$  also.



...

# Min-Norm Point and SFM

... proof of Thm. 19.6.1 cont.

- Then  $(x^* + \alpha \mathbf{1}_e - \alpha \mathbf{1}_{e'})(E)$   
 $= x^*(E \setminus \{e, e'\}) + \underbrace{(x^*(e) + \alpha)}_{x_{\text{new}}^*(e)} + \underbrace{(x^*(e') - \alpha)}_{x_{\text{new}}^*(e')} = f(E).$

# Min-Norm Point and SFM

... proof of Thm. 19.6.1 cont.

- Then  $(x^* + \alpha \mathbf{1}_e - \alpha \mathbf{1}_{e'})(E)$   
 $= x^*(E \setminus \{e, e'\}) + \underbrace{(x^*(e) + \alpha)}_{x_{\text{new}}^*(e)} + \underbrace{(x^*(e') - \alpha)}_{x_{\text{new}}^*(e')} = f(E).$
- Minimality of  $x^* \in B_f$  in  $l_2$  sense requires that, with such an  $\alpha > 0$ ,  
 $(x^*(e))^2 + (x^*(e'))^2 < (x_{\text{new}}^*(e))^2 + (x_{\text{new}}^*(e'))^2$



# Min-Norm Point and SFM

... proof of Thm. 19.6.1 cont.

- Then  $(x^* + \alpha \mathbf{1}_e - \alpha \mathbf{1}_{e'})(E)$   
 $= x^*(E \setminus \{e, e'\}) + \underbrace{(x^*(e) + \alpha)}_{x_{\text{new}}^*(e)} + \underbrace{(x^*(e') - \alpha)}_{x_{\text{new}}^*(e')} = f(E).$
- Minimality of  $x^* \in B_f$  in  $l_2$  sense requires that, with such an  $\alpha > 0$ ,  
 $(x^*(e))^2 + (x^*(e'))^2 < (x_{\text{new}}^*(e))^2 + (x_{\text{new}}^*(e'))^2$
- Given that  $e \in A_-$ ,  $x^*(e) < 0$ . Thus, if  $x^*(e') > 0$ , we would have  
 $(x^*(e) + \alpha')^2 + (x^*(e') - \alpha')^2 < (x^*(e))^2 + (x^*(e'))^2$ , for some  
 $0 < \alpha' \leq \alpha$ , contradicting the optimality of  $x^*$ .

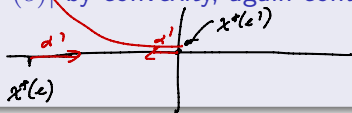


$$\therefore x^*(e') \leq 0$$

# Min-Norm Point and SFM

... proof of Thm. 19.6.1 cont.

- Then  $(x^* + \alpha \mathbf{1}_e - \alpha \mathbf{1}_{e'})(E)$   
 $= x^*(E \setminus \{e, e'\}) + \underbrace{(x^*(e) + \alpha)}_{x_{\text{new}}^*(e)} + \underbrace{(x^*(e') - \alpha)}_{x_{\text{new}}^*(e')} = f(E).$
- Minimality of  $x^* \in B_f$  in  $l_2$  sense requires that, with such an  $\alpha > 0$ ,  
 $(x^*(e))^2 + (x^*(e'))^2 < (x_{\text{new}}^*(e))^2 + (x_{\text{new}}^*(e'))^2$
- Given that  $e \in A_-$ ,  $x^*(e) < 0$ . Thus, if  $x^*(e') > 0$ , we would have  
 $(x^*(e) + \alpha')^2 + (x^*(e') - \alpha')^2 < (x^*(e))^2 + (x^*(e'))^2$ , for some  
 $0 < \alpha' \leq \alpha$ , contradicting the optimality of  $x^*$ .
- If  $x^*(e') = 0$ , we would have  $(x^*(e) + \alpha')^2 + (\alpha')^2 < (x^*(e))^2$ , for any  
 $0 < \alpha' < |x^*(e)|$  by convexity, again contradicting optimality of  $x^*$ .



$$\therefore x^*(e') < 0$$

# Min-Norm Point and SFM

... proof of Thm. 19.6.1 cont.

- Then  $(x^* + \alpha \mathbf{1}_e - \alpha \mathbf{1}_{e'})(E)$   
 $= x^*(E \setminus \{e, e'\}) + \underbrace{(x^*(e) + \alpha)}_{x_{\text{new}}^*(e)} + \underbrace{(x^*(e') - \alpha)}_{x_{\text{new}}^*(e')} = f(E).$
- Minimality of  $x^* \in B_f$  in  $l_2$  sense requires that, with such an  $\alpha > 0$ ,  
 $(x^*(e))^2 + (x^*(e'))^2 < (x_{\text{new}}^*(e))^2 + (x_{\text{new}}^*(e'))^2$
- Given that  $e \in A_-$ ,  $x^*(e) < 0$ . Thus, if  $x^*(e') > 0$ , we would have  
 $(x^*(e) + \alpha')^2 + (x^*(e') - \alpha')^2 < (x^*(e))^2 + (x^*(e'))^2$ , for some  
 $0 < \alpha' \leq \alpha$ , contradicting the optimality of  $x^*$ .
- If  $x^*(e') = 0$ , we would have  $(x^*(e) + \alpha')^2 + (\alpha')^2 < (x^*(e))^2$ , for any  
 $0 < \alpha' < |x^*(e)|$  by convexity, again contradicting optimality of  $x^*$ .
- Thus, we must have  $x^*(e') < 0$  (strict negativity).

...

# Min-Norm Point and SFM

... proof of Thm. 19.6.1 cont.

- Thus, for a pair  $(e, e')$  with  $e' \in \text{dep}(x^*, e)$  and  $e \in A_-$ , we have  $x(e') < 0$  and hence  $e' \in A_-$ .

...

# Min-Norm Point and SFM

... proof of Thm. 19.6.1 cont.

- Thus, for a pair  $(e, e')$  with  $e' \in \text{dep}(x^*, e)$  and  $e \in A_-$ , we have  $x(e') < 0$  and hence  $e' \in A_-$ .
- Hence,  $\forall e \in A_-$ , we have  $\text{dep}(x^*, e) \subseteq A_-$ .

...

# Min-Norm Point and SFM

... proof of Thm. 19.6.1 cont.

- Thus, for a pair  $(e, e')$  with  $e' \in \text{dep}(x^*, e)$  and  $e \in A_-$ , we have  $x(e') < 0$  and hence  $e' \in A_-$ .
- Hence,  $\forall e \in A_-$ , we have  $\text{dep}(x^*, e) \subseteq A_-$ .
- A very similar argument can show that,  $\forall e \in A_0$ , we have  $\text{dep}(x^*, e) \subseteq A_0$ .

...

# Min-Norm Point and SFM

... proof of Thm. 19.6.1 cont.

- Thus, for a pair  $(e, e')$  with  $e' \in \text{dep}(x^*, e)$  and  $e \in A_-$ , we have  $x(e') < 0$  and hence  $e' \in A_-$ .
- Hence,  $\forall e \in A_-$ , we have  $\text{dep}(x^*, e) \subseteq A_-$ .
- A very similar argument can show that,  $\forall e \in A_0$ , we have  $\text{dep}(x^*, e) \subseteq A_0$ .
- Also, recall that  $e \in \text{dep}(x^*, e)$ .

...

# Min-Norm Point and SFM

... proof of Thm. 19.6.1 cont.

- Therefore, we have  $\cup_{e \in A_-} \text{dep}(x^*, e) = A_-$  and  $\cup_{e \in A_0} \text{dep}(x^*, e) = A_0$



# Min-Norm Point and SFM

... proof of Thm. 19.6.1 cont.

- Therefore, we have  $\cup_{e \in A_-} \text{dep}(x^*, e) = A_-$  and  $\cup_{e \in A_0} \text{dep}(x^*, e) = A_0$
- i.e.,  $\{\text{dep}(x^*, e)\}_{e \in A_-}$  is cover for  $A_-$ , as is  $\{\text{dep}(x^*, e)\}_{e \in A_0}$  for  $A_0$ .

# Min-Norm Point and SFM

... proof of Thm. 19.6.1 cont.

- Therefore, we have  $\cup_{e \in A_-} \text{dep}(x^*, e) = A_-$  and  $\cup_{e \in A_0} \text{dep}(x^*, e) = A_0$
- i.e.,  $\{\text{dep}(x^*, e)\}_{e \in A_-}$  is cover for  $A_-$ , as is  $\{\text{dep}(x^*, e)\}_{e \in A_0}$  for  $A_0$ .
- $\text{dep}(x^*, e)$  is minimal tight set containing  $e$ , meaning  $x^*(\text{dep}(x^*, e)) = f(\text{dep}(x^*, e))$ , and since tight sets are closed under union, we have that  $A_-$  and  $A_0$  are also tight, meaning:

# Min-Norm Point and SFM

... proof of Thm. 19.6.1 cont.

- Therefore, we have  $\cup_{e \in A_-} \text{dep}(x^*, e) = A_-$  and  $\cup_{e \in A_0} \text{dep}(x^*, e) = A_0$
- i.e.,  $\{\text{dep}(x^*, e)\}_{e \in A_-}$  is cover for  $A_-$ , as is  $\{\text{dep}(x^*, e)\}_{e \in A_0}$  for  $A_0$ .
- $\text{dep}(x^*, e)$  is minimal tight set containing  $e$ , meaning  $x^*(\text{dep}(x^*, e)) = f(\text{dep}(x^*, e))$ , and since tight sets are closed under union, we have that  $A_-$  and  $A_0$  are also tight, meaning:

$$x^*(A_-) = f(A_-) \tag{19.28}$$

# Min-Norm Point and SFM

... proof of Thm. 19.6.1 cont.

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- Hence,  $f(A_-) = f(A_0)$ , meaning  $A_-$  and  $A_0$  have the same valuation, but we have not yet shown they are the minimizers of the submodular function, nor that they are, resp. the maximal and minimal minimizers.



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- Hence,  $y^*$  is a maximizer of l.h.s. of Eqn. (19.7), and  $A_-$  and  $A_0$  are minimizers of  $f$ .

# Min-Norm Point and SFM

... proof of Thm. 19.6.1 cont.

- We next show that, not only are they minimizers, but  $A_-$  is the unique minimal and  $A_0$  is the unique maximal minimizer of  $f$





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- Hence,  $A_-$  must be the unique minimal minimizer of  $f$ , and  $A_0$  is the unique maximal minimizer of  $f$ .



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- This is currently the best practical algorithm for **general purpose** submodular function minimization.
- **But its underlying lower-bound strong poly complexity is unknown.**

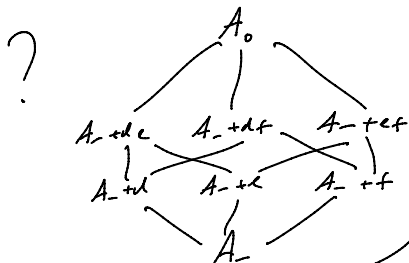
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No.  
not  
nec.



$$A_- = \{a, b, c\}$$

$$A_0 = \{d, e, f\} \cup A_-$$

$$A_- = \{a, b, c\}$$

$$A_0 = A_- \cup R \cup S \cup T$$

$$R \cap S = \emptyset, \quad S \cap T = \emptyset, \quad T \cap R = \emptyset$$

$$|R| > 1 \quad |S| > 1 \quad |T| > 1$$

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## Theorem 19.6.2

Let  $A \subseteq E$  be **any** minimizer of submodular  $f$ , and let  $x^*$  be the minimum-norm point. Then  $A$  can be expressed in the form:

$$A = A_- \cup \bigcup_{a \in A_m} \text{dep}(x^*, a) \quad (19.34)$$

for some set  $A_m \subseteq A_0 \setminus A_-$ . Conversely, for any set  $A_m \subseteq A_0 \setminus A_-$ , then  $A \triangleq A_- \cup \bigcup_{a \in A_m} \text{dep}(x^*, a)$  is a minimizer.

# Min-norm point and other minimizers of $f$

proof of Thm. 19.6.2.

- If  $A$  is a minimizer, then  $A_- \subseteq A \subseteq A_0$ , and  $f(A) = y^*(E)$  is the minimum valuation of  $f$ .

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- But  $x^* \in P_f$ , so  $x^*(A) \leq f(A)$  and  $f(A) = x^*(A_-) \leq x^*(A)$ .
- Also, since  $A \subseteq A_0$  and  $x^*(A_0 \setminus A) = 0$ ,  $x^*(A_-) = x^*(A) = x^*(A_0)$





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- For any  $a \in A$ ,  $A$  is a tight set containing  $a$ , and  $\text{dep}(x^*, a)$  is the minimal tight containing  $a$ .

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- Hence, for any  $a \in A$ ,  $\text{dep}(x^*, a) \subseteq A$ .
- This means that  $\bigcup_{a \in A} \text{dep}(x^*, a) = A$ .

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- Hence, for any  $a \in A$ ,  $\text{dep}(x^*, a) \subseteq A$ .
- This means that  $\bigcup_{a \in A} \text{dep}(x^*, a) = A$ .
- Since  $A_- \subseteq A \subseteq A_0$ , then  $\exists A_m \subseteq A \setminus A_-$  such that

$$A = \bigcup_{a \in A_-} \text{dep}(x^*, a) \cup \bigcup_{a \in A_m} \text{dep}(x^*, a) = A_- \cup \bigcup_{a \in A_m} \text{dep}(x^*, a)$$

...

# Min-norm point and other minimizers of $f$

proof of Thm. 19.6.2.

- Conversely, consider any set  $A_m \subseteq A_0 \setminus A_-$ , and define  $A$  as

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- Then since  $A$  is a union of tight sets,  $A$  is also a tight set, and we have  $f(A) = x^*(A)$ .



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- Then since  $A$  is a union of tight sets,  $A$  is also a tight set, and we have  $f(A) = x^*(A)$ .
- But  $x^*(A \setminus A_-) = 0$ , so  $f(A) = x^*(A) = x^*(A_-) = f(A_-)$  meaning  $A$  is also a minimizer of  $f$ .





# Min-norm point and other minimizers of $f$

## proof of Thm. 19.6.2.

- Conversely, consider any set  $A_m \subseteq A_0 \setminus A_-$ , and define  $A$  as

$$A = A_- \cup \bigcup_{a \in A_m} \text{dep}(x^*, a) = \bigcup_{a \in A_-} \text{dep}(x^*, a) \cup \bigcup_{a \in A_m} \text{dep}(x^*, a) \quad (19.35)$$

- Then since  $A$  is a union of tight sets,  $A$  is also a tight set, and we have  $f(A) = x^*(A)$ .
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Therefore, we can generate the entire lattice of minimizers of  $f$  starting from  $A_-$  and  $A_0$  given access to  $\text{dep}(x^*, e)$ .

# On a unique minimizer $f$

- Note that if  $f(e|A) > 0$ ,  $\forall A \subseteq E$  and  $e \in E \setminus A$ , then we have  $A_- = A_0$  (there is one unique minimizer).

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- On the other hand, if  $A_- = A_0$ , it does not imply  $f(e|A) > 0$  for all  $A \subseteq E \setminus \{e\}$ .
- If  $A_- = A_0$  then certainly  $f(e|A_0) > 0$  for  $e \in E \setminus A_0$  and  $-f(e|A_0 \setminus \{e\}) > 0$  for all  $e \in A_0$ .

# Duality: convex minimization of L.E. and min-norm alg.

- Let  $f$  be a submodular function with  $\tilde{f}$  its Lovász extension. Then the following two problems are duals (Bach-2013):

$$\underset{w \in \mathbb{R}^V}{\text{minimize}} \quad \tilde{f}(w) + \frac{1}{2} \|w\|_2^2 \quad (19.36)$$

$$\text{maximize} \quad - \|x\|_2^2 \quad (19.37a)$$

$$\text{subject to} \quad x \in B_f \quad (19.37b)$$

where  $B_f = P_f \cap \{x \in \mathbb{R}^V : x(V) = f(V)\}$  is the base polytope of submodular function  $f$ , and  $\|x\|_2^2 = \sum_{e \in V} x(e)^2$  is squared 2-norm.

- Equation (19.36) is related to proximal methods to minimize the Lovász extension (see Parikh&Boyd, "Proximal Algorithms" 2013).
- Equation (19.37b) is solved by the minimum-norm point algorithm (Wolfe-1976, Fujishige-1984, Fujishige-2005, Fujishige-2011) is (as we will see) essentially an active-set procedure for quadratic programming, and uses Edmonds's greedy algorithm to make it efficient.
- Unknown strongly poly worst-case running time, although in practice it usually performs quite well (see below).

# Convex and affine hulls, affinely independent

- Given points set  $P = \{p_1, p_2, \dots, p_k\}$  with  $p_i \in \mathbb{R}^V$ , let  $\text{conv } P$  be the **convex hull of  $P$** , i.e.,

$$\text{conv } P \triangleq \left\{ \sum_{i=1}^k \lambda_i p_i : \sum_i \lambda_i = 1, \lambda_i \geq 0, i \in [k] \right\}. \quad (19.38)$$

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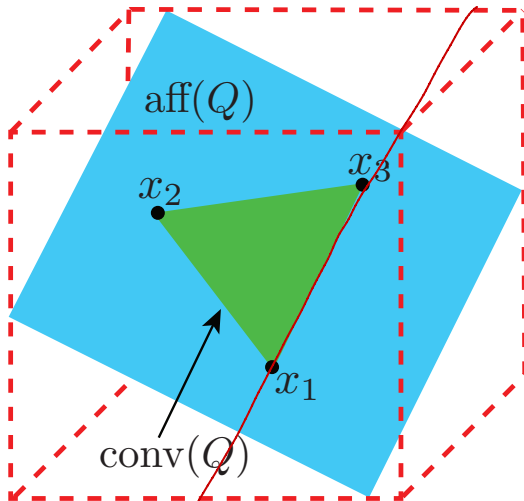
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- A set of points  $Q$  is **affinely independent** if no point in  $Q$  *belongs* to the affine hull of the remaining points.

# Convex vs. Affine hull, geometry



$$\forall i, x_i \in \mathbb{R}^3$$

$$Q = \{x_1, x_2, x_3\}$$

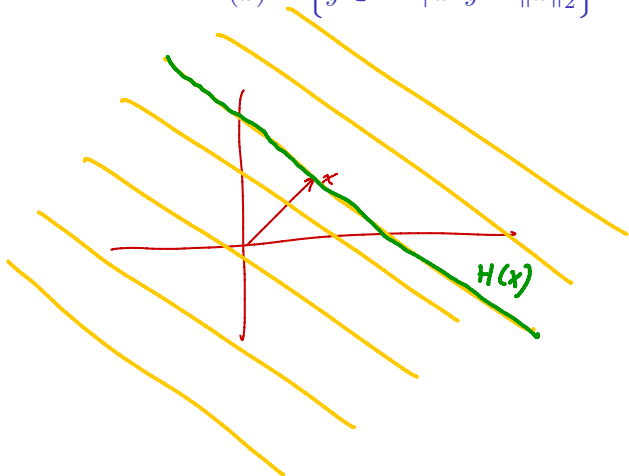
$x_1, x_2, x_3$  coplanar

←  $\text{span}(Q)$

# $H(x)$ : Orthogonal $x$ -containing hyperplane

- Define  $H(x)$  as the hyperplane that is orthogonal to the line from 0 to  $x$ , while also containing  $x$ , i.e.

$$H(x) \triangleq \left\{ y \in \mathbb{R}^V \mid x^\top y = \|x\|_2^2 \right\} \quad (19.40)$$

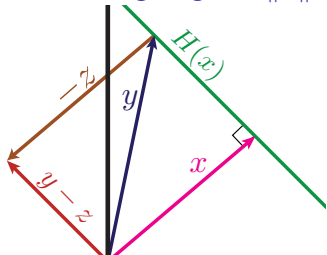


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- Any set  $\{y \in \mathbb{R}^V \mid x^\top y = c\}$  is orthogonal to the line from 0 to  $x$ . This follows since, for constant  $z$ ,  $\{y : (y - z)^\top x = 0\} = \{y : y^\top x = z^\top x\}$  is hyperplane orthogonal to  $x$  translated by  $z$ . Take  $c = z^\top x$  for result, and  $z = x$ , giving  $c = \|x\|^2$ , to contain  $x$ .

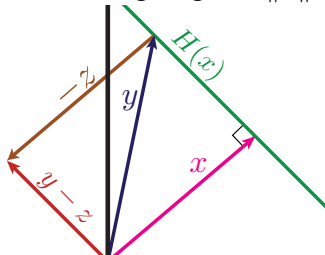


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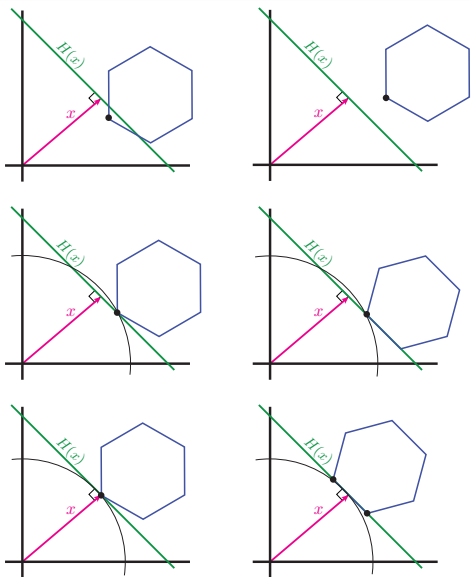
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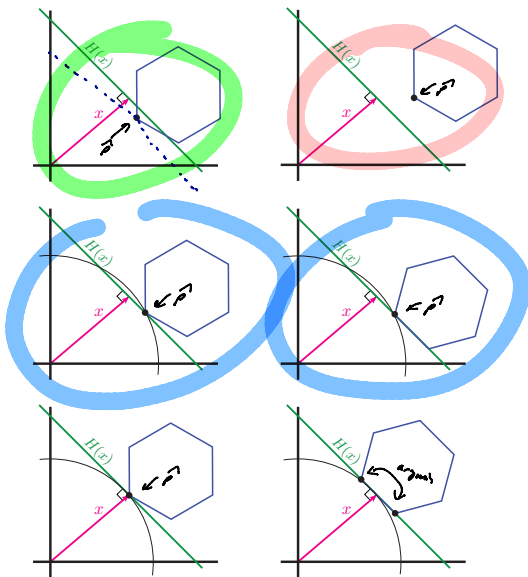
# Ex: $H(x)$ , polytopes, and supporting hyperplanes

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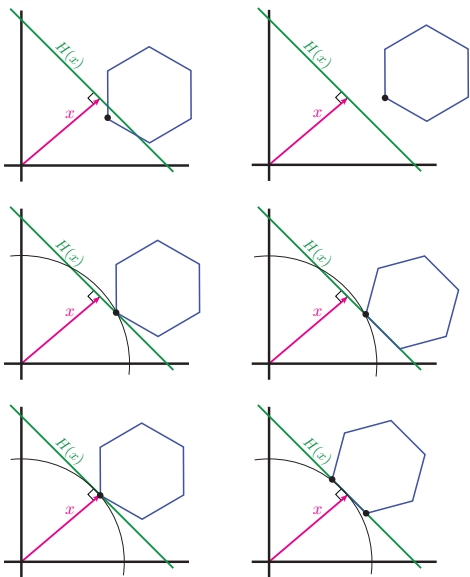
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- Consider conv  $P$  polytope for points  $P = \{p_1, p_2, \dots\}$ , and  $\hat{p} \in \operatorname{argmin}_{p \in P} x^\top p$ . TL:  $x^\top \hat{p} < x^\top x$ ; TR:  $x^\top \hat{p} > x^\top x$ ; middle row:  $x^\top \hat{p} = x^\top x$ .





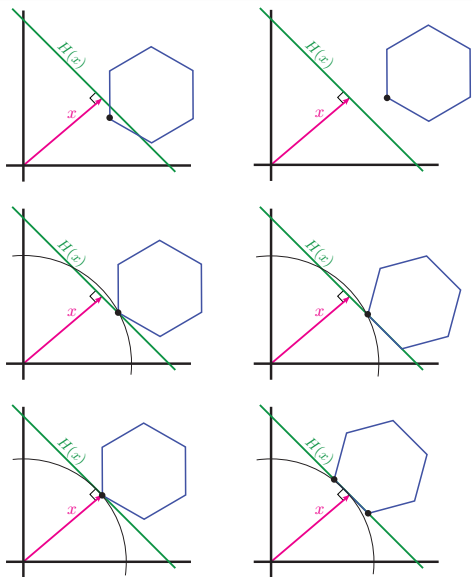
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- Bottom Row: In Algo,  $x$  is chosen so that if  $x^\top \hat{p} = x^\top x$  then  $H(x)$  separates  $P$  from the origin, and  $x$  is the min 2-norm point. Notice that  $x^\top p \geq x^\top x$  for all  $p \in P$ .



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- Middle/bottom row:  $H(x)$  is a **supporting hyperplane** of  $\text{conv } P$  (contained, touching).



# Notation

- The line between  $x$  and  $y$ : given two points  $x, y \in \mathbb{R}^V$ , let  $[x, y] \triangleq \{\lambda x + (1-\lambda)y : \lambda \in [0, 1]\}$ . Hence,  $[x, y] = \text{conv} \{x, y\}$ .

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- Note, if we wish to minimize the 2-norm of a vector  $\|x\|_2$ , we can equivalently minimize its square  $\|x\|_2^2 = \sum_i x_i^2$ , and vice versa.

# Frank-Wolfe vs. Fujishige-Wolfe

An algorithm we will not use for the min-norm is M. Frank & P. Wolfe “An algorithm for quadratic programming”, 1956, or conditional gradient descent for constrained convex minimization given convex function  $f : \mathcal{D} \rightarrow \mathbb{R}$ .

---

**Input** : Convex  $f : \mathcal{D} \rightarrow \mathbb{R}$ ,  $x_0 \in \mathcal{D}$

**Output**:  $x^* \in \mathcal{D}$ , the minimizer of  $f$ .

- 1  $k \leftarrow 0$  and start with  $x_0 \in \mathcal{D}$  ;
  - 2 Let  $s_k$  solve  $\min \langle s, \nabla f(x_k) \rangle$  s.t.  $s \in \mathcal{D}$  ;
  - 3 Let  $\lambda_k \in [0, 1]$  minimize  $f(\lambda s_k + (1 - \lambda)x_k)$  ;
  - 4  $x_{k+1} \leftarrow \lambda_k s_k + (1 - \lambda_k)x_k$ ,  $k \leftarrow k + 1$  ;
  - 5 Goto line 1 if  $\|x_{k+1} - x_k\| > \tau$  ;
  - 6  $x^* \leftarrow x_{k+1}$
- 

- Above could minimize Lovász extension, primal approach to SFM.
- For finding the min-norm point, we will be using the P. Wolfe, “Finding the Nearest Point in a Polytope”, 1976 which is the same Wolfe but different algorithm and different year.

# Fujishige-Wolfe Min-Norm Algorithm

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- Seems to still be (among) the fastest general purpose SFM algo.
- Algorithm maintains a set of points  $Q \subseteq P$ , which is always assuredly *affinely independent*.  $Q$  doesn't grow big.

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- When  $Q$  are affinely independent, minimum norm point in the affine hull of  $Q$  can easily be found, as a closed form solution for  $\min_{x \in \text{aff } Q} \|x\|_2$  is available (see below).

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- If number of extreme points is exponential, hard to do in general.
- Number of extreme points of submodular base polytope is exponentially large, but linear optimization over the base polytope  $B_f$  doable  $O(n \log n)$  time via Edmonds's greedy algorithm.

# Pseudocode of Fujishige-Wolfe Min-Norm (MN) algorithm

**Input** :  $P = \{p_1, \dots, p_m\}, p_i \in \mathbb{R}^n, i = 1, \dots, m.$

**Output**:  $x^*$ : the minimum-norm-point in  $\text{conv } P.$

```
1  $x^* \leftarrow p_{i^*}$  where  $p_{i^*} \in \text{argmin}_{p \in P} \|p\|_2$  /* or choose it arbitrarily */ ;
2  $Q \leftarrow \{x^*\};$ 
3 while 1 do /* major loop */
4   if  $x^* = 0$  or  $H(x^*)$  separates  $P$  from origin then
5     | return :  $x^*$ 
6   else
7     | Choose  $\hat{x} \in P$  on the near (closer to 0) side of  $H(x^*)$ ;
8     |  $Q = Q \cup \{\hat{x}\};$ 
9   while 1 do /* minor loop */
10    |  $x_0 \leftarrow \text{argmin}_{x \in \text{aff } Q} \|x\|_2;$ 
11    | if  $x_0 \in \text{conv } Q$  then
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# Fujishige-Wolfe Min-Norm algorithm: Geometric Example

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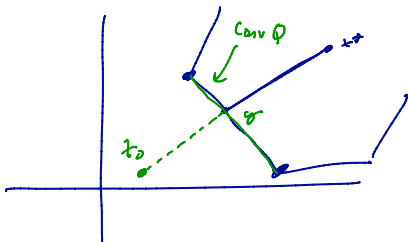
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$$\underline{x^* \in \text{conv } Q \subseteq \text{conv } P}, \quad (19.42)$$

must hold at every possible assignment of  $x^*$  (Lines 1, 11, and 16):

- 1 True after Line 1 since  $Q = \{x^*\}$ ,
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- There are six places that might be seemingly tricky or expensive: Line 4, Line 6, Line 9, Line 10, Line 14, and Line 15.
- We will consider each in turn, but first we do a geometric example.

# Pseudocode of Fujishige-Wolfe Min-Norm (MN) algorithm

**Input** :  $P = \{p_1, \dots, p_m\}, p_i \in \mathbb{R}^n, i = 1, \dots, m.$

**Output**:  $x^*$ : the minimum-norm-point in  $\text{conv } P.$

```
1  $x^* \leftarrow p_{i^*}$  where  $p_{i^*} \in \text{argmin}_{p \in P} \|p\|_2$  /* or choose it arbitrarily */ ;
2  $Q \leftarrow \{x^*\};$ 
3 while 1 do /* major loop */
4   if  $x^* = 0$  or  $H(x^*)$  separates  $P$  from origin then
5     return :  $x^*$ 
6   else
7     Choose  $\hat{x} \in P$  on the near (closer to 0) side of  $H(x^*)$ ;
8      $Q = Q \cup \{\hat{x}\};$ 
9   while 1 do /* minor loop */
10     $x_0 \leftarrow \text{argmin}_{x \in \text{aff } Q} \|x\|_2;$ 
11    if  $x_0 \in \text{conv } Q$  then
12       $x^* \leftarrow x_0;$ 
13      break;
14    else
15       $y \leftarrow \text{argmin}_{x \in \text{conv } Q \cap [x^*, x_0]} \|x - x_0\|_2;$ 
16      Delete from  $Q$  points not on the face of  $\text{conv } Q$  where  $y$  lies;
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```

Solved by Edmond's greedy procedure.

Solved via linear equation solver.

Linear equation solver represents  $x_0$  as affine coeffs, so this just checks  $\geq 0$ .

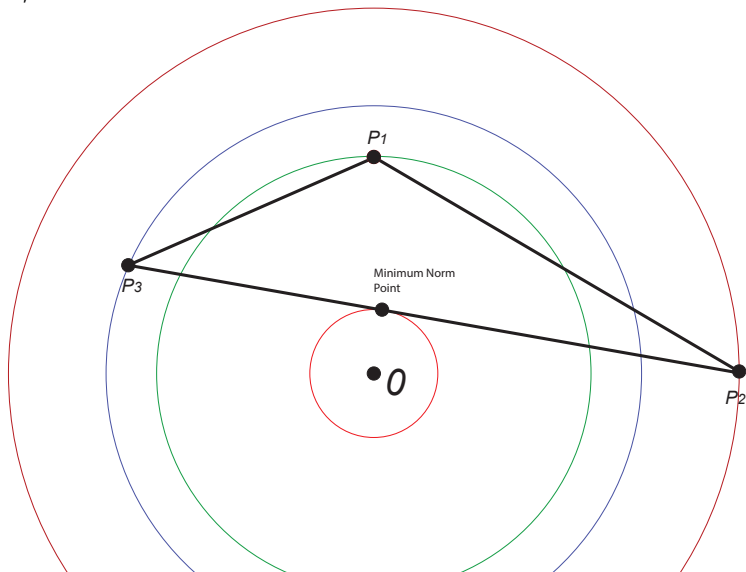
Doable since we're representing points as convex combinations of points within  $Q$

# Fujishige-Wolfe Min-Norm algorithm: Geometric Example

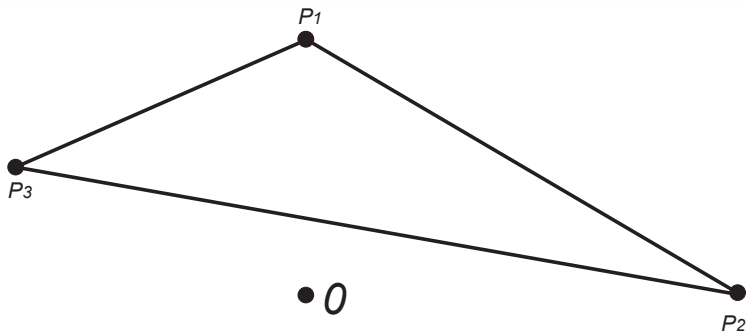
- In the following series of images, permanent (non-changing) named points on the polytope will be indicated by capital letters (i.e.,  $P_1, P_2, P_3, R, S, T$ ) while variables in the algorithm that are changing will use lower case letters (i.e.,  $x^*, x_0, \hat{x}, y$ ).
- Also, example is in 2D, so polytope given can't be a real base  $B_f$  for any  $f$ . Example meant to show only the geometry of the algorithm.

# Fujishige-Wolfe Min-Norm algorithm: Geometric Example

Polytope, and circles concentric at 0.

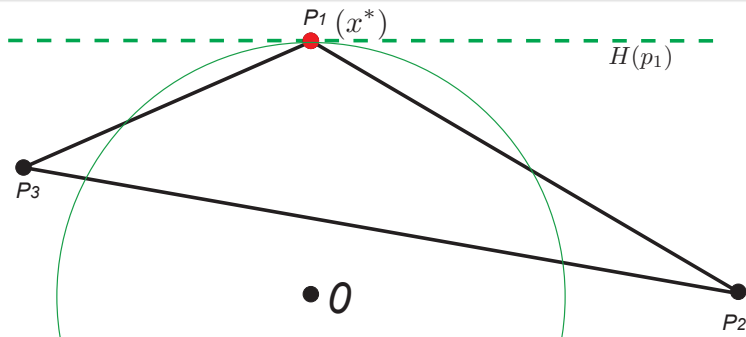


# Fujishige-Wolfe Min-Norm algorithm: Geometric Example



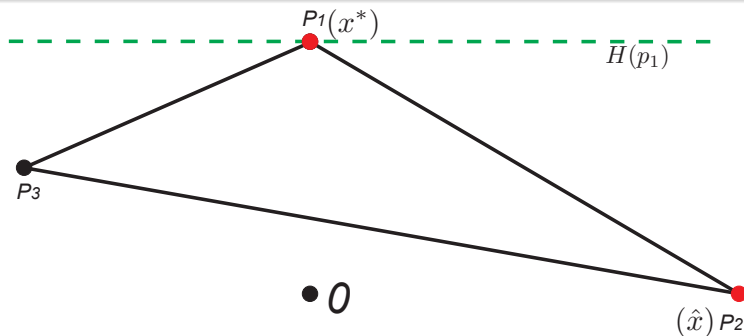
The initial polytope consisting of the convex hull of three points  $p_1, p_2, p_3$ , and the origin  $0$ .

# Fujishige-Wolfe Min-Norm algorithm: Geometric Example



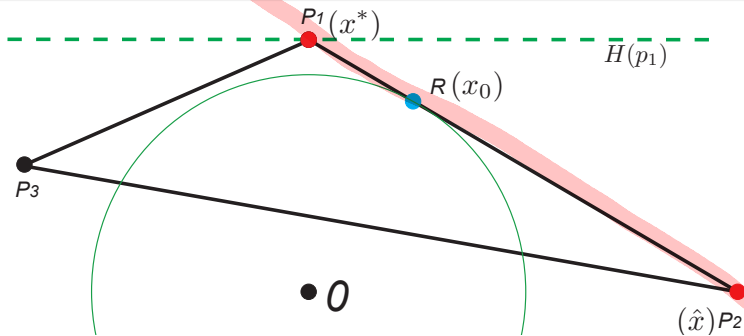
$p_1$  is the extreme point closest to  $0$  and so we choose it first, although we can choose any arbitrary extreme point as the initial point. We set  $x^* \leftarrow p_1$  in Line 1, and  $Q \leftarrow \{p_1\}$  in Line 2.  $H(x^*) = H(p_1)$  (green dashed line) is not a supporting hyperplane of  $\text{conv}(P)$  in Line 4, so we move on to the else condition in Line 5.

# Fujishige-Wolfe Min-Norm algorithm: Geometric Example



We need to add some extreme point  $\hat{x}$  on the “near” side of  $H(p_1)$  in Line 6, we choose  $\hat{x} = p_2$ . In Line 7, we set  $Q \leftarrow Q \cup \{p_2\}$ , so  $Q = \{p_1, p_2\}$ .

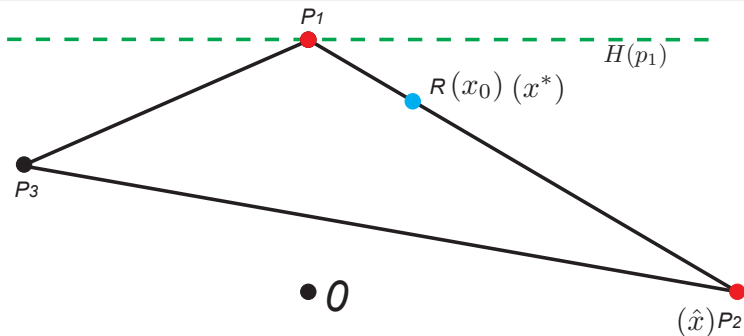
# Fujishige-Wolfe Min-Norm algorithm: Geometric Example



$x_0 = R$  is the min-norm point in  $\text{aff} \{p_1, p_2\}$  computed in Line 9.

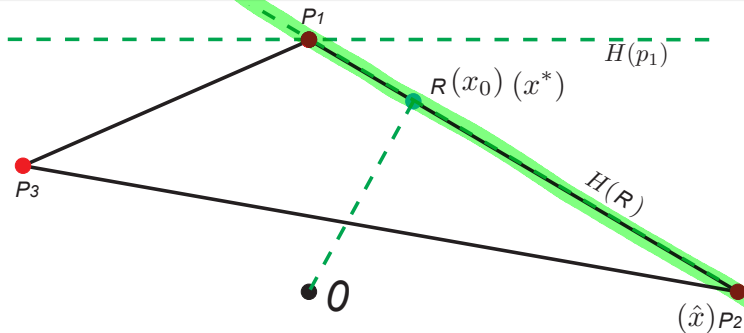


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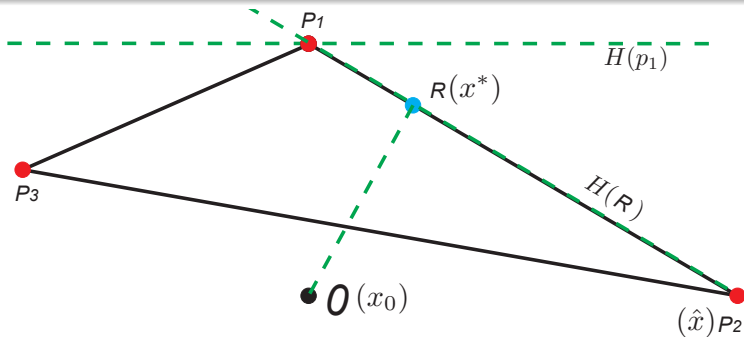
$x_0 = R$  is the min-norm point in  $\text{aff} \{p_1, p_2\}$  computed in Line 9. Also, with  $Q = \{p_1, p_2\}$ , since  $R \in \text{conv} Q$ , we set  $x^* \leftarrow x_0 = R$  in Line 11, not violating the invariant  $x^* \in \text{conv} Q$ . Note, after Line 11, we still have  $x^* \in \text{conv} P$  and  $\|x^*\|_2 = \|x_{\text{new}}^*\|_2 < \|x_{\text{old}}^*\|_2$  strictly.

# Fujishige-Wolfe Min-Norm algorithm: Geometric Example



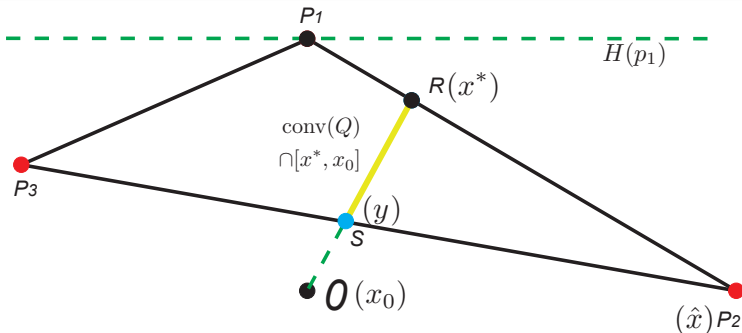
$R = x_0 = x^*$ . We consider next  $H(R) = H(x^*)$  in Line 4.  $H(x^*)$  is not a supporting hyperplane of  $\text{conv } P$ . So we choose  $p_3$  on the “near” side of  $H(x^*)$  in Line 6. Add  $Q \leftarrow Q \cup \{p_3\}$  in Line 7. Now  $Q = P = \{p_1, p_2, p_3\}$ .

# Fujishige-Wolfe Min-Norm algorithm: Geometric Example



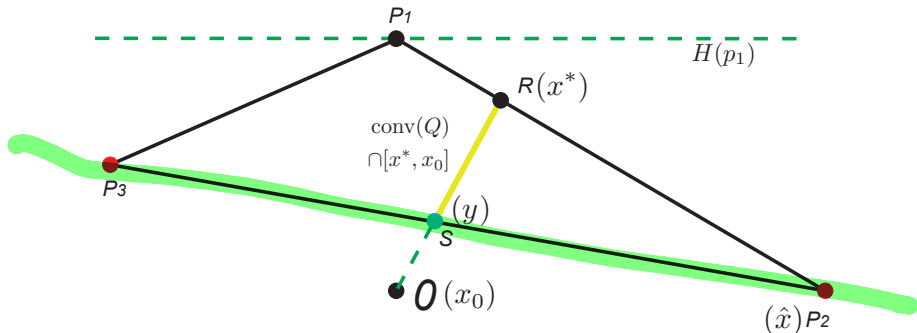
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# Fujishige-Wolfe Min-Norm algorithm: Geometric Example



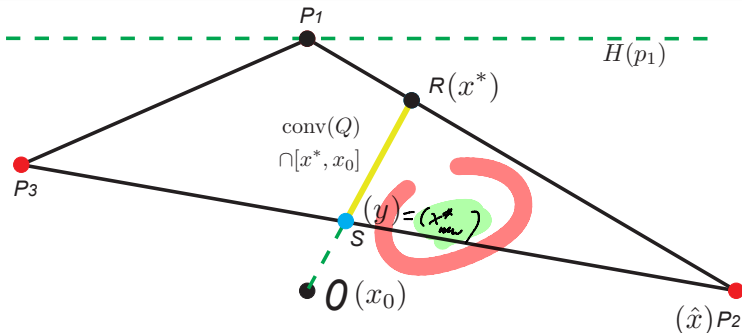
$Q = P = \{p_1, p_2, p_3\}$ . Line 14:  $S = y = \operatorname{argmin}_{x \in \text{conv } Q \cap [x^*, x_0]} \|x - x_0\|_2$  where  $x_0$  is 0 and  $x^*$  is  $R$  here. Thus,  $y$  lies on the boundary of  $\text{conv } Q$ . Note,  $\|y\|_2 < \|x^*\|_2$  since  $x^* \in \text{conv } Q$ ,  $\|x_0\|_2 < \|x^*\|_2$ .

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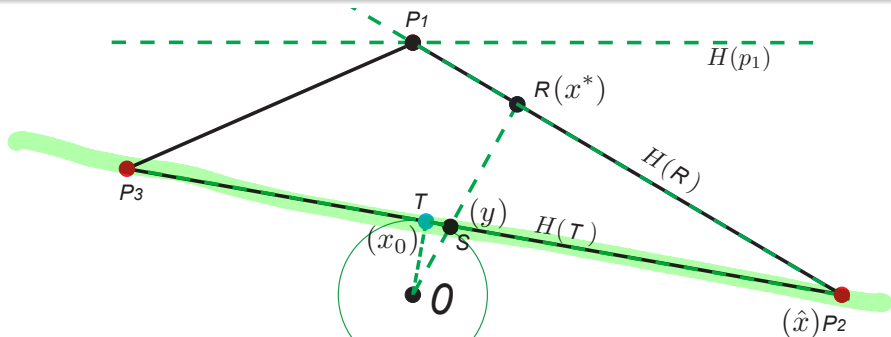
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# Fujishige-Wolfe Min-Norm algorithm: Geometric Example



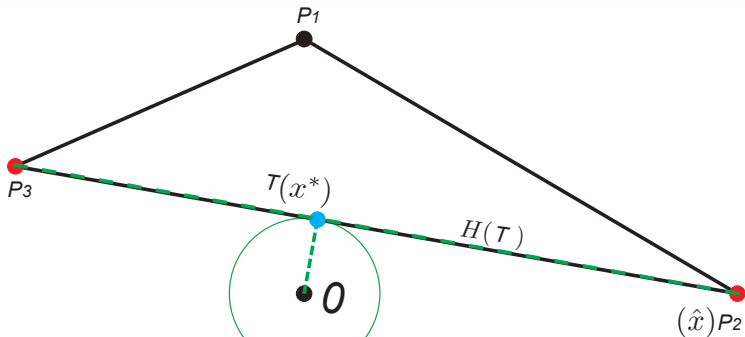
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# Fujishige-Wolfe Min-Norm algorithm: Geometric Example



$Q = \{p_2, p_3\}$ , and so  $x_0 = T$  computed in Line 9 is the min-norm point in  $\text{aff } Q$ . We also have  $x_0 \in \text{conv } Q$  in Line 10 so we assign  $x^* \leftarrow x_0$  in Line 11 and break.

# Fujishige-Wolfe Min-Norm algorithm: Geometric Example



$H(T)$  separates  $P$  from the origin in Line 4, and therefore is a supporting hyperplane, and therefore  $x^*$  is the min-norm point in  $\text{conv } P$ , so we return with  $x^*$ .



# Condition for Min-Norm Point

## Theorem 19.7.1

$P = \{p_1, p_2, \dots, p_m\}$ ,  $x^* \in \text{conv } P$  is the min. norm point in  $\text{conv } P$  iff

$$p_i^\top x^* \geq \|x^*\|_2^2 \quad \forall i = 1, \dots, m. \quad (19.44)$$

## Proof.

- Assume  $x^*$  is the min-norm point, let  $y \in \text{conv } P$ , and  $0 \leq \theta \leq 1$ .

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$$= \|x^*\|_2^2 + 2\theta(x^{*\top}y - x^{*\top}x^*) + \theta^2 \|y - x^*\|_2^2 \quad (19.46)$$

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- Conversely, given Eq (19.44), and given that  $y = \sum_i \lambda_i p_i \in \text{conv } P$ ,

$$y^\top x^* = \sum_i \lambda_i p_i^\top x^* \geq \sum_i \lambda_i x^{*\top} x^* = x^{*\top} x^* \quad (19.47)$$

implying that  $\|z\|_2^2 > \|x^*\|_2^2$  in Equation 19.46 for arbitrary  $z \in \text{conv } P$ . □

The set  $Q$  is always affinely independent

### Lemma 19.7.2

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- $\therefore$  update  $Q \cup \{\hat{x}\}$  at Line 7 is affinely independent as long as  $Q$  is. □

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- Since  $\hat{x} \notin H(x^*)$  chosen at Line 6, we have  $\hat{x} \notin \text{aff } Q$ .
- $\therefore$  update  $Q \cup \{\hat{x}\}$  at Line 7 is affinely independent as long as  $Q$  is. □

Thus, by Lemma 19.7.2, we have for any  $x \in \text{aff } Q$  such that  $x = \sum_i w_i q_i$  with  $\sum_i w_i = 1$ , the weights  $w_i$  are uniquely determined.

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## Proof.

This is immediate, since  $Q$  is always affinely independent, and in  $\mathbb{R}^V$ , an affinely independent set can have at most  $n + 1$  entries, with  $|V| = n$ .  $\square$

# Minimum Norm in an affine set

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- $k + 1$  variables and  $k$  unknowns, solvable with linear solver with matrices

$$\begin{bmatrix} 0 & \mathbf{1}^T \\ \mathbf{1} & Q^T Q \end{bmatrix} \begin{bmatrix} \lambda \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix} \quad (19.52)$$

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- Thanks to  $Q$  being affine, matrix on l.h.s. is invertable.

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- Note, this also solves Line 10, since feasibility requires  $\sum_i w_i = 1$ , we need only check  $w \geq 0$  to ensure  $x_0 = \sum_i w_i q_i \in \text{conv } Q$ .

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- Given  $w$  and  $v$ , we can also easily solve Lines 14 and 15 (see "Step 3" on page 133 of Wolfe-1976, which also defines numerical tolerances).



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- Given  $w$  and  $v$ , we can also easily solve Lines 14 and 15 (see "Step 3" on page 133 of Wolfe-1976, which also defines numerical tolerances).
- We have yet to see how to efficiently solve Lines 4 and 6, however.

# MN Algorithm finds the MN point in finite time.

## Theorem 19.7.4

*The MN Algorithm finds the minimum norm point in  $\text{conv } P$  after a finite number of iterations of the major loop.*

## Proof.

- In minor loop, we always have  $x^* \in \text{conv } Q$ , since whenever  $Q$  is modified,  $x^*$  is updated as well (Line 16) such that the updated  $x^*$  remains in new  $\text{conv } Q$ .

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- Hence, every time  $x^*$  is updated (in minor loop), its norm never increases, i.e., before Line 11,  $\|x_0\|_2 \leq \|x^*\|_2$  since  $x^* \in \text{aff } Q$  and  $x_0 = \min_{x \in \text{aff } Q} \|x\|_2$ . Similarly, before Line 16,  $\|y\|_2 \leq \|x^*\|_2$ , since invariant  $x^* \in \text{conv } Q$  but while  $x_0 \in \text{aff } Q$ , we have  $x_0 \notin \text{conv } Q$ , and  $\|x_0\|_2 < \|x^*\|_2$ .

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- Moreover, there can be no more iterations within a minor loop than the dimension of  $\text{conv } Q$  for the initial  $Q$  given to the minor loop initially at Line 8 (dimension of  $\text{conv } Q$  is  $|Q| - 1$  since  $Q$  is affinely independent).

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- When  $Q$  reduces to a singleton, the minor loop always terminates.
- Thus, the minor loop terminates in finite number of iterations, at most dimension of  $Q$ .
- In fact, total number of iterations of minor loop in entire algorithm is at most number of points in  $P$  since we never add back in points to  $Q$  that have been removed.

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- Each time  $Q$  is augmented with  $\hat{x}$  at Line 7, followed by updating  $x^*$  with  $x_0$  at Line 11, (i.e., when the minor loop returns with only one iteration),  $\|x^*\|_2$  strictly decreases from what it was before.

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- To see this, consider  $x^* + \theta(\hat{x} - x^*)$  where  $0 \leq \theta \leq 1$ . Since both  $\hat{x}, x^* \in \text{conv } Q$ , we have  $x^* + \theta(\hat{x} - x^*) \in \text{conv } Q$ .

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- Therefore, we have  $\|x^* + \theta(\hat{x} - x^*)\|_2 \geq \|x_0\|_2$ , which implies

$$\begin{aligned} \|x^* + \theta(\hat{x} - x^*)\|_2^2 &= \|x^*\|_2^2 + 2\theta \left( (x^*)^\top \hat{x} - \|x^*\|_2^2 \right) + \theta^2 \|\hat{x} - x^*\|_2^2 \\ &\geq \|x_0\|_2^2 \end{aligned} \quad (19.53)$$

and from Line 6,  $\hat{x}$  is on the same side of  $H(x^*)$  as the origin, i.e.  $(x^*)^\top \hat{x} < \|x^*\|_2^2$ , so middle term of r.h.s. of equality is negative.

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- Therefore, for sufficiently small  $\theta$ , specifically for

$$\theta < \frac{2 \left( \|x^*\|_2^2 - (x^*)^\top \hat{x} \right)}{\|\hat{x} - x^*\|_2^2} \quad (19.54)$$

we have that  $\|x^*\|_2^2 > \|x_0\|_2^2$ .



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- For a similar reason, we have  $\|x^*\|_2$  strictly decreases each time  $Q$  is updated at Line 7 and followed by updating  $x^*$  with  $y$  at Line 16.



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- For a similar reason, we have  $\|x^*\|_2$  strictly decreases each time  $Q$  is updated at Line 7 and followed by updating  $x^*$  with  $y$  at Line 16.
- Therefore, in each iteration of major loop,  $\|x^*\|_2$  strictly decreases, and the MN Algorithm must terminate and it can only do so when the optimal is found.





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- When  $0 \leq \theta < \frac{2(\|x^*\|_2^2 - (x^*)^\top \hat{x})}{\|\hat{x} - x^*\|_2^2}$ , we can get the maximal value of the lower bound, over  $\theta$ , as follows:

$$\max_{0 \leq \theta < \frac{2(\|x^*\|_2^2 - (x^*)^\top \hat{x})}{\|\hat{x} - x^*\|_2^2}} \underline{\Delta} = \left( \frac{\|x^*\|_2^2 - (x^*)^\top \hat{x}}{\|\hat{x} - x^*\|_2} \right)^2 \quad (19.56)$$

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- To maximize lower bound of norm reduction at each major iteration, want to find an  $\hat{x}$  such that the above lower bound (Equation 19.56) is maximized.

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- This problem, however, is at least as hard as the MN problem itself as we have a quadratic term in the denominator.

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- As a surrogate, we maximize numerator in Eqn. 19.57, i.e., find

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- Also, solution  $\hat{x}$  in Line 6 can be used to determine if hyperplane  $H(x^*)$  separates  $\operatorname{conv} P$  from the origin (Line 4): if the point in  $P$  having greatest distance to  $H(x^*)$  is not on the side where origin lies, then  $H(x^*)$  separates  $\operatorname{conv} P$  from the origin.

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- Mathematically and theoretically, we terminate the algorithm if

$$(x^*)^\top \hat{x} \geq \|x^*\|_2^2, \quad (19.59)$$

where  $\hat{x}$  is the solution of Eq. 19.58.

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- In practice, the above optimality test might never hold numerically. Hence, as suggested by Wolfe, we introduce a tolerance parameter  $\epsilon > 0$ , and terminates the algorithm if

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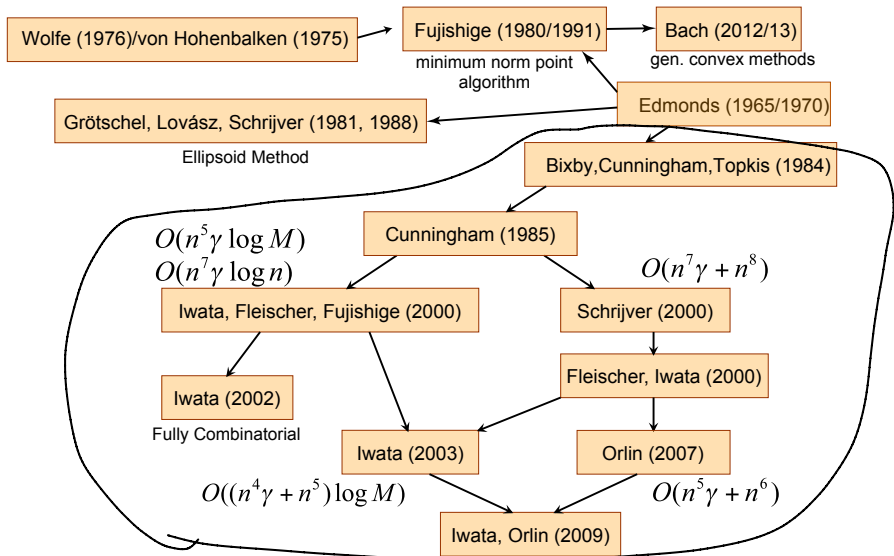
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- Edmond's greedy algorithm, therefore, solves both Line 4 and Line 6 simultaneously.
- Hence, Edmonds's discovery is one of the main reasons that the MN algorithm is applicable to submodular function minimization.

# SFM Summary (modified from S. Iwata's slides)

## General Submodular Function Minimization





# MN Algorithm Complexity

- The currently fastest strongly polynomial combinatorial algorithm for SFM achieves a running time of  $O(n^5T + n^6)$  (Orlin'09) where  $T$  is the time for function evaluation, far from practical for large problem instances.

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- Since the number of major iterations required is unknown, the complexity of MN is also unknown.



# MN Algorithm Empirical Complexity

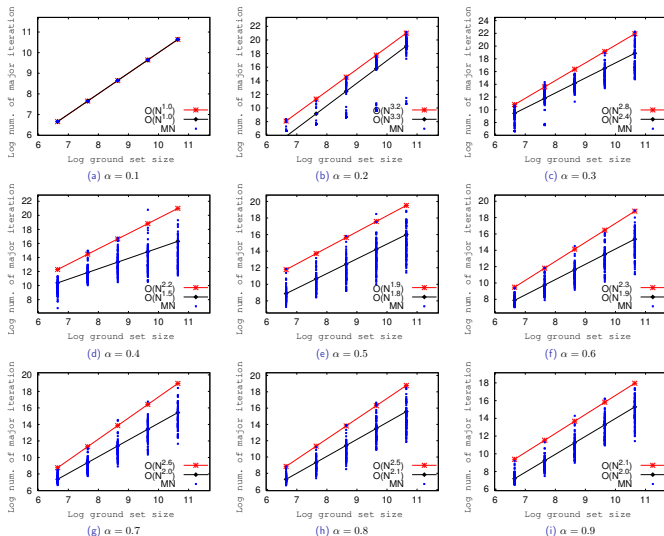


Figure: The number of major iteration for  $f(S) = -m_1(S) + 100 \cdot (w_1(\mathcal{N}(S)))^\alpha$ . The red lines are the linear interpolations of the worst case points, and the black lines are the linear interpolations of the average case points. From Lin&Bilmes 2014 (unpublished)

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- This is pseudo-polynomial since it depends on the function values.
- There currently is no known polynomial time complexity analysis for this algorithm.