Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture 19 —

http://www.ee.washington.edu/people/faculty/bilmes/classes/ee563_spring_2018/

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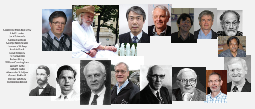
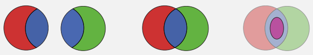
June 6th, 2018

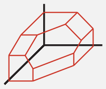


Clockwise from top left:v
László Lovász
Jack Edmonds
Satoru Fujishige
George Nemhauser
Laurence Wolsey
András Frank
Lloyd Shapley
H. Narayanan
Robert Bixby
William Cunningham
William Tutte
Richard Rado
Alexander Schrijver
Garrett Birkhoff
Hassler Whitney
Richard Dedekind

$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$

$= f(A_r) + 2f(C) + f(B_r)$ $= f(A_r) + f(C) + f(B_r)$ $= f(A \cap B)$





Announcements, Assignments, and Reminders

- Take home final exam (like long homework). Due Friday, June 8th, 4:00pm via our assignment dropbox (https://canvas.uw.edu/courses/1216339/assignments).
- Get started now. At least read through everything and ask any questions you might have.
- As always, if you have any questions about anything, please ask then via our discussion board (https://canvas.uw.edu/courses/1216339/discussion_topics). Can meet at odd hours via zoom (send message on canvas to schedule time to chat).

Class Road Map - EE563

- L1(3/26): Motivation, Applications, & Basic Definitions,
- L2(3/28): Machine Learning Apps (diversity, complexity, parameter, learning target, surrogate).
- L3(4/2): Info theory exs, more apps, definitions, graph/combinatorial examples
- L4(4/4): Graph and Combinatorial Examples, Matrix Rank, Examples and Properties, visualizations
- L5(4/9): More Examples/Properties/ Other Submodular Defs., Independence,
- L6(4/11): Matroids, Matroid Examples, Matroid Rank, Partition/Laminar Matroids
- L7(4/16): Laminar Matroids, System of Distinct Reps, Transversals, Transversal Matroid, Matroid Representation, Dual Matroids
- L8(4/18): Dual Matroids, Other Matroid Properties, Combinatorial Geometries, Matroids and Greedy.
- L9(4/23): Polyhedra, Matroid Polytopes, Matroids → Polymatroids
- L10(4/29): Matroids → Polymatroids, Polymatroids, Polymatroids and Greedy,
- L11(4/30): Polymatroids, Polymatroids and Greedy
- L12(5/2): Polymatroids and Greedy, Extreme Points, Cardinality Constrained Maximization
- L13(5/7): Constrained Submodular Maximization
- L14(5/9): Submodular Max w. Other Constraints, Cont. Extensions, Lovasz Extension
- L15(5/14): Cont. Extensions, Lovasz Extension, Choquet Integration, Properties
- L16(5/16): More Lovasz extension, Choquet, defs/props, examples, multiliear extension
- L17(5/21): Finish L.E., Multilinear Extension, Submodular Max/polyhedral approaches, Most Violated inequality, Still More on Matroids, Closure/Sat
- L–(5/28): Memorial Day (holiday)
- L18(5/30): Closure/Sat, Fund. Circuit/Dep
- L19(6/6): Fund. Circuit/Dep, Min-Norm Point Definitions, Review & Support for Min-Norm, Proof that min-norm gives optimal, Computing Min-Norm Vector for B_f maximization.

Last day of instruction, June 1st. Finals Week: June 2-8, 2018.

Summary of Concepts

- Most violated inequality $\max \{x(A) - f(A) : A \subseteq E\}$
- Matroid by circuits, and the fundamental circuit $C(I, e) \subseteq I + e$.
- Minimizers of submodular functions form a lattice.
- Minimal and maximal element of a lattice.
- x-tight sets, maximal and minimal tight set.
- sat function & Closure
- Saturation Capacity
- e-containing tight sets
- dep function & fundamental circuit of a matroid

Summary important definitions so far: tight, dep, & sat

- x-tight sets: For $x \in P_f$, $\mathcal{D}(x) \triangleq \{A \subseteq E : x(A) = f(A)\}$.
- Polymatroid closure/maximal x-tight set: For $x \in P_f$, $\text{sat}(x) \triangleq \cup\{A : A \in \mathcal{D}(x)\} = \{e : e \in E, \forall \alpha > 0, x + \alpha \mathbf{1}_e \notin P_f\}$.
- Saturation capacity: for $x \in P_f$, $0 \leq \hat{c}(x;e) \triangleq \min\{f(A) - x(A) | \forall A \ni e\} = \max\{\alpha : \alpha \in \mathbb{R}, x + \alpha \mathbf{1}_e \in P_f\}$
- Recall: $\text{sat}(x) = \{e : \hat{c}(x;e) = 0\}$ and $E \setminus \text{sat}(x) = \{e : \hat{c}(x;e) > 0\}$.
- e-containing x-tight sets: For $x \in P_f$, $\mathcal{D}(x,e) = \{A : e \in A \subseteq E, x(A) = f(A)\} \subseteq \mathcal{D}(x)$.
- Minimal e-containing x-tight set/polymatroidal fundamental circuit/: For $x \in P_f$,

$$\text{dep}(x,e) = \begin{cases} \bigcap\{A : e \in A \subseteq E, x(A) = f(A)\} & \text{if } e \in \text{sat}(x) \\ \emptyset & \text{else} \end{cases}$$

$$= \{e' : \exists \alpha > 0, \text{ s.t. } x + \alpha(\mathbf{1}_e - \mathbf{1}_{e'}) \in P_f\}$$

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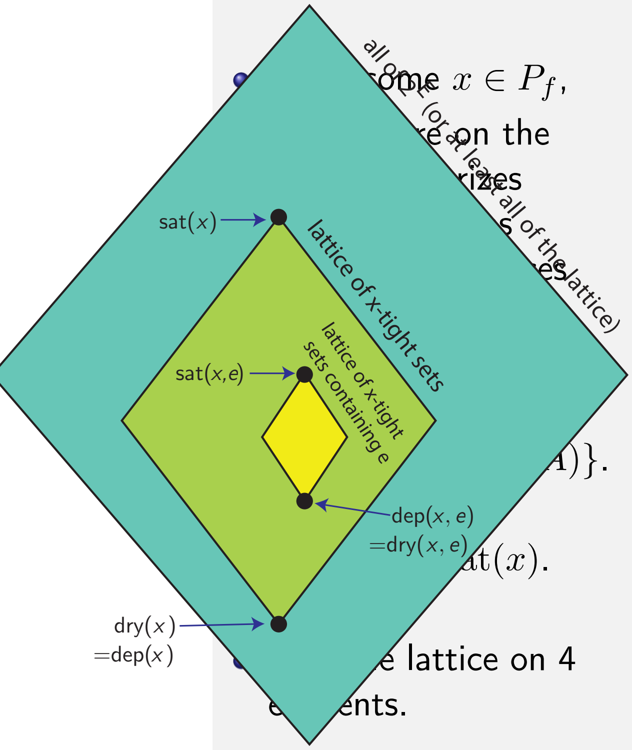
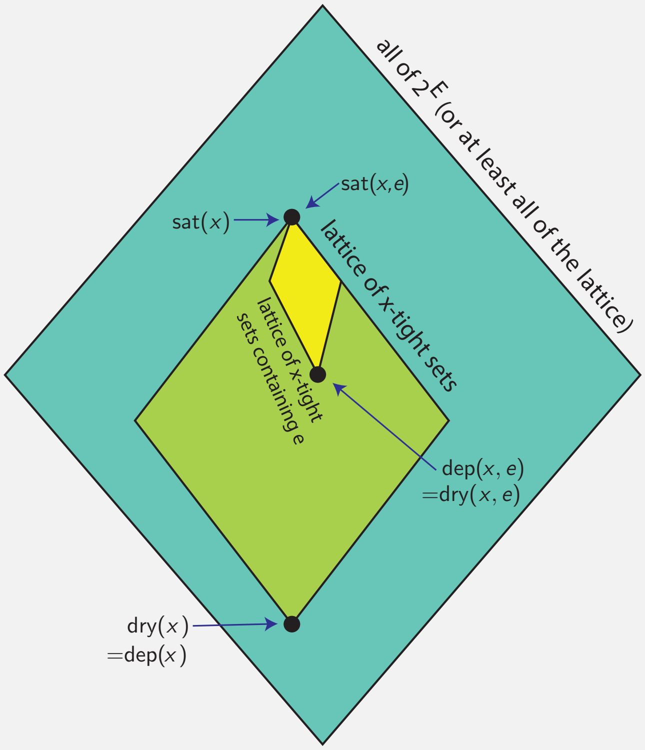


all of 2E (or at least all of the lattice)
sat(x)
lattice of x-tight sets
sat(x,e)
lattice of x-tight sets containing e
dep(x, e)
=dry(x, e)
dry(x)
=dep(x)



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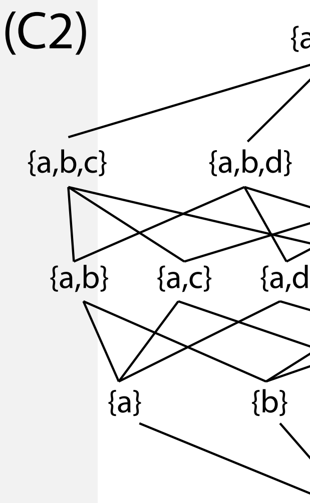

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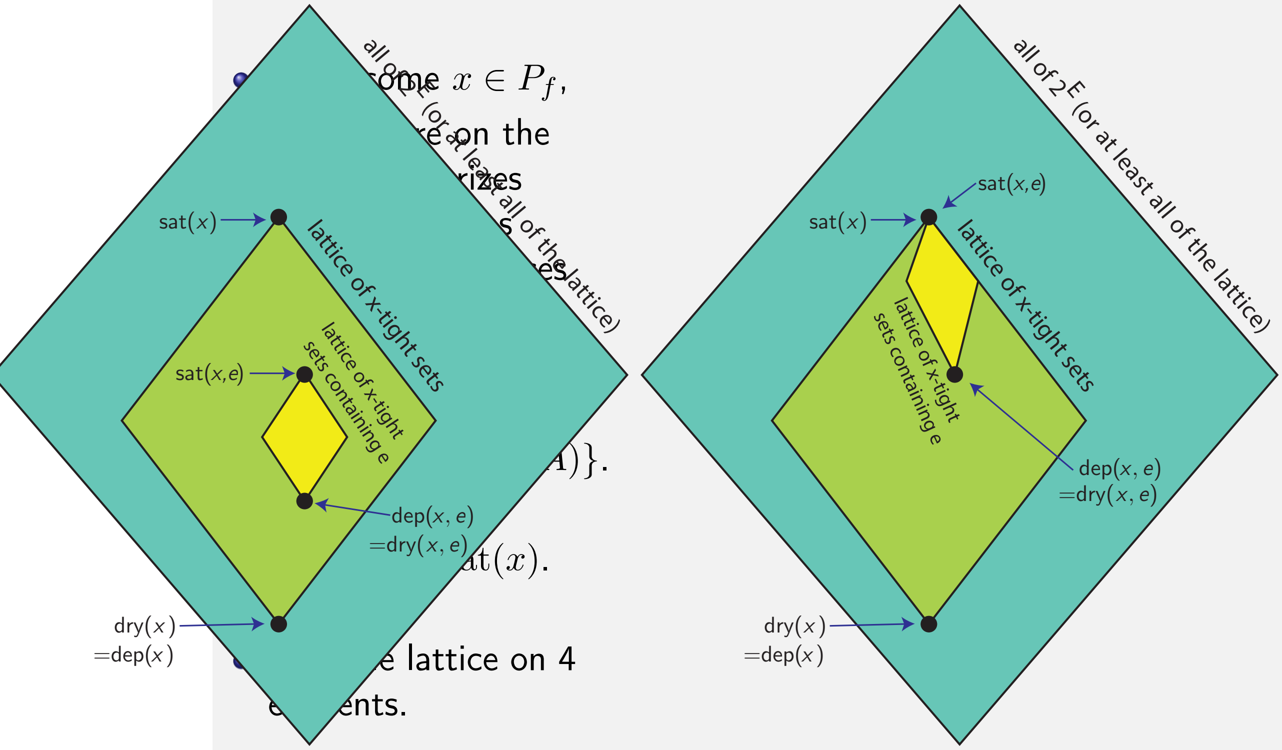
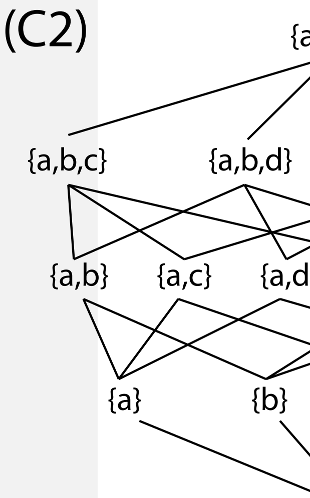




Submodular Function Minimization (SFM) and Min-Norm

- We saw that SFM can be used to solve most violated inequality problems for a given $x \in P_f$ and, in general, SFM can solve the question "Is $x \in P_f$" by seeing if x violates any inequality (if the most violated one is negative, solution to SFM, then $x \in P_f$).
- Unconstrained SFM, $\min_{A \subseteq V} f(A)$ solves many other problems as well in combinatorial optimization, machine learning, and other fields.
- We next study an algorithm, the "Fujishige-Wolfe Algorithm", or what is known as the "Minimum Norm Point" algorithm, which is an active set method to do this, and one that in practice works about as well as anything else people (so far) have tried for general purpose SFM.
- Note special case SFM can be much faster.

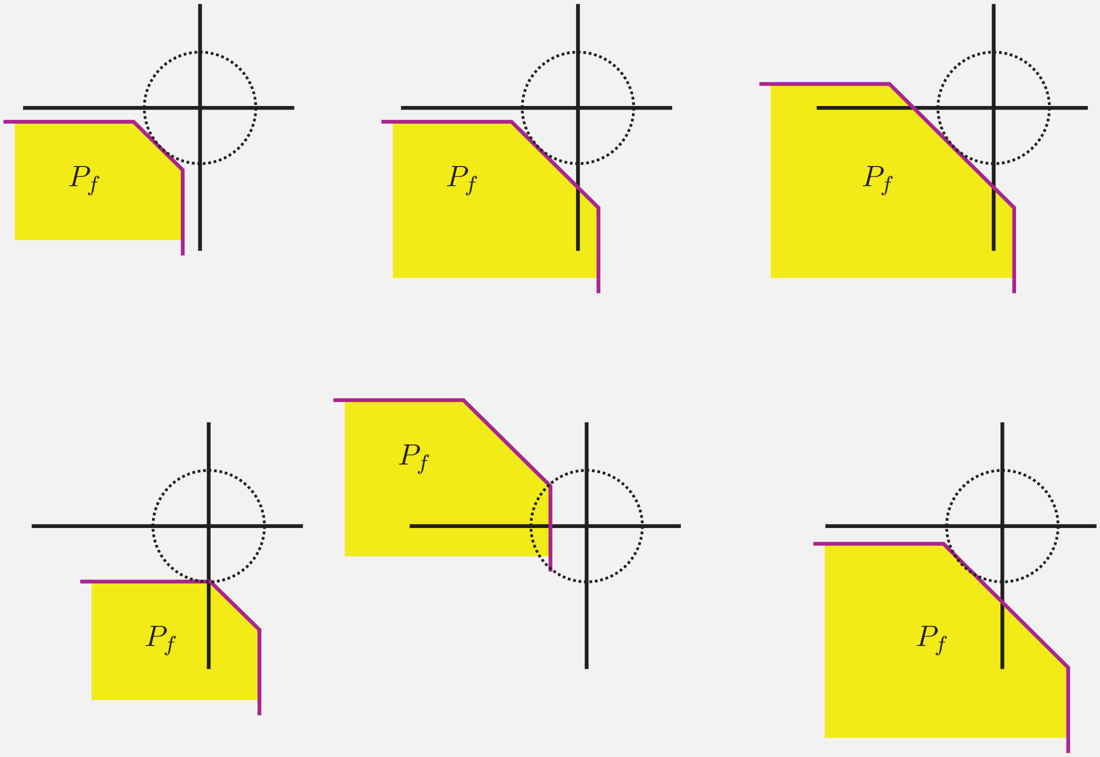
Min-Norm Point: Definition

- Consider the optimization:

$$\begin{aligned} &\text{minimize} && \|x\|_2^2 && \text{(19.1a)}\\ &\text{subject to} && x \in B_f && \text{(19.1b)} \end{aligned}$$

  where B_f is the base polytope of submodular f, and $\|x\|_2^2 = \sum_{e \in E} x(e)^2$ is the squared 2-norm. Let x^* be the optimal solution.
- Note, x^* is the unique optimal solution since we have a strictly convex objective over a set of convex constraints.
- x^* is called the minimum norm point of the base polytope.

Min-Norm Point: Examples

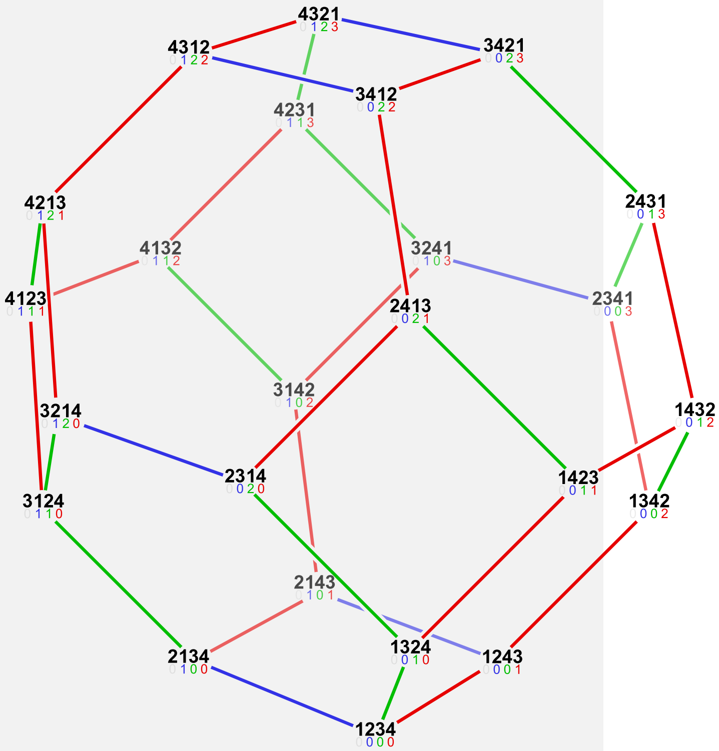


Ex: 3D base B_f: permutahedron

- Consider submodular function $f : 2^V \to \mathbb{R}$ with $n = |V| = 4$, and for $X \subseteq V$, concave g,

$$f(X) = g(|X|) = \sum_{i=1}^{|X|}(n - i + 1)$$
$$= |X|\Big(n - \frac{|X| - 1}{2}\Big)$$

- Then B_f is a 3D polytope, and in this particular case gives us a permutahedron with 24 distinct extreme points, on the right (from wikipedia).



Min-Norm Point and Submodular Function Minimization

- Given optimal solution x^* to $[\min \|x\|_2^2$ s.t. $x \in B_f]$, and consider:

$$y^* = x^* \wedge 0 = (\min(x^*(e), 0) | e \in E), \tag{19.2}$$
$$A_- = \{e : x^*(e) < 0\}, \tag{19.3}$$
$$A_0 = \{e : x^*(e) \leq 0\}. \tag{19.4}$$

- Thus, we immediately have that:

$$A_- \subseteq A_0 \tag{19.5}$$

and that

$$x^*(A_-) = x^*(A_0) = y^*(A_-) = y^*(A_0) \tag{19.6}$$

- It turns out, these quantities will solve the submodular function minimization problem, as we now show.
- The proof is nice since it uses the tools we've been recently developing.

More about the base B_f

Theorem 19.5.1

Let f be a polymatroid function and suppose that E can be partitioned into $(E_1, E_2, \dots, E_k)$ such that $f(A) = \sum_{i=1}^{k} f(A \cap E_i)$ for all $A \subseteq E$, and k is maximum. Then the base polytope $B_f = \{x \in P_f : x(E) = f(E)\}$ (the E-tight subset of P_f) has dimension $|E| - k$.

- In fact, every $x \in P_f$ is dominated by $x \leq y \in B_f$.

Theorem 19.5.2

If $x \in P_f$ and T is tight for x (meaning $x(T) = f(T)$), then there exists $y \in B_f$ with $x \leq y$ and $y(e) = x(e)$ for $e \in T$.

- We leave the proof as an exercise.

Review from Lecture 12

The following slide repeats Theorem 12.3.2 from lecture 12 and is one of the most important theorems in submodular theory.

A polymatroid function's polyhedron is a polymatroid.

Theorem 19.5.1

Let f be a submodular function defined on subsets of E. For any $x \in \mathbb{R}^E$, we have:

$$\textit{rank}(x) = \max\left(y(E) : y \leq x, y \in P_f\right) = \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \tag{19.1}$$

Essentially the same theorem as Theorem 10.4.1, but note P_f rather than P_f^+. Taking $x = 0$ we get:

Corollary 19.5.2

Let f be a submodular function defined on subsets of E. We have:

$$\textit{rank}(0) = \max\left(y(E) : y \leq 0, y \in P_f\right) = \min\left(f(A) : A \subseteq E\right) \tag{19.2}$$

Modified max-min theorem

- Min-max theorem (Thm 12.3.2) restated for $x = 0$.

$$\max\left\{y(E) | y \in P_f, y \leq 0\right\} = \min\left\{f(X) | X \subseteq V\right\} \tag{19.7}$$

Theorem 19.5.3 (Edmonds-1970)

$$\min\left\{f(X) | X \subseteq E\right\} = \max\left\{x^-(E) | x \in B_f\right\} \tag{19.8}$$

where $x^-(e) = \min\left\{x(e), 0\right\}$ *for* $e \in E$.

Proof via the Lovász ext.

$$\min\left\{f(X) | X \subseteq E\right\} = \min_{w \in [0,1]^E} \tilde{f}(w) = \min_{w \in [0,1]^E} \max_{x \in P_f} w^\intercal x \tag{19.9}$$

$$= \min_{w \in [0,1]^E} \max_{x \in B_f} w^\intercal x \tag{19.10}$$

$$= \max_{x \in B_f} \min_{w \in [0,1]^E} w^\intercal x \tag{19.11}$$

$$= \max_{x \in B_f} x^-(E) \tag{19.12}$$

□

Convexity, Strong duality, and min/max swap

The min/max switch follows from strong duality. I.e., consider $g(w, x) = w^{\intercal}x$ and we have domains $w \in [0, 1]^E$ and $x \in B_f$. then for any $(w, x) \in [0, 1]^E \times B_f$, we have

$$\min_{w' \in [0,1]^E} g(w', x) \leq g(w, x) \leq \max_{x' \in B_f} g(w, x') \tag{19.13}$$

which means that we have weak duality

$$\max_{x \in B_f} \min_{w' \in [0,1]^E} g(w', x) \leq \min_{w \in [0,1]^E} \max_{x' \in B_f} g(w, x') \tag{19.14}$$

but since $g(w, x)$ is linear, we have strong duality, meaning

$$\max_{x \in B_f} \min_{w' \in [0,1]^E} g(w', x) = \min_{w \in [0,1]^E} \max_{x' \in B_f} g(w, x') \tag{19.15}$$

Alternate proof of modified max-min theorem

We start directly from Theorem 12.3.2.

$$\max\left(y(E) : y \leq 0, y \in P_f\right) = \min\left(f(A) : A \subseteq E\right) \tag{19.16}$$

Given $y \in \mathbb{R}^E$, define $y^- \in \mathbb{R}^E$ with $y^-(e) = \min\{y(e), 0\}$ for $e \in E$.

$$\begin{aligned} \max\left(y(E) : y \leq 0, y \in P_f\right) &= \max\left(y^-(E) : y \leq 0, y \in P_f\right) && (19.17)\\ &= \max\left(y^-(E) : y \in P_f\right) && (19.18)\\ &= \max\left(y^-(E) : y \in B_f\right) && (19.19) \end{aligned}$$

The first equality follows since $y \leq 0$. The second equality (together with the first) shown on following slide. The third equality follows since for any $x \in P_f$ there exists a $y \in B_f$ with $x \leq y$ (follows from Theorem 19.5.2).

Alternate proof of modified max-min theorem

Consider the following two problems:

$$\max \sum_{e \in E} y(e) \quad (19.20a)$$
$$\text{s.t. } y \leq x \quad (19.20b)$$
$$y \in P \quad (19.20c)$$

$$\max \sum_{e \in E} \min(y(e), x(e)) \quad (19.21a)$$
$$\text{s.t. } y \in P \quad (19.21b)$$

- Solutions identical cost. Let y_1^* be l.h.s. OPT and y_2^* be r.h.s. OPT.
- Consider y_1^* as r.h.s. solution and suppose it is worse than r.h.s. OPT:
$$\sum_{e \in E} \min(y_1^*(e), x(e)) < \sum_{e \in E} \min(y_2^*(e), x(e)) \quad (19.22)$$
- Hence, $\exists e'$ s.t. $y_1^*(e') < \min(y_2^*(e'), x(e'))$. Recall $y_1^*, y_2^* \in P$.
- This implies $\sum_{e \neq e'} y_1^*(e) + y_1^*(e') < \sum_{e \neq e'} y_1^*(e) + \min(y_2^*(e'), x(e'))$, better feasible solution to l.h.s., contradicting y_1^*'s optimality for l.h.s.
- Similarly, consider y_2^* as l.h.s. solution, suppose worse than l.h.s. OPT
$$\sum_{e \in E} y_2^*(e) < \sum_{e \in E} y_1^*(e) \quad (19.23)$$
- Then $\exists e'$ such that $y_2^*(e') < y_1^*(e') \leq x(e')$.

- This implies that replacing $y_2(e')$'s value with $y_1(e')$ is still feasible for r.h.s. but better, contradicting y_2^*'s optimality.
- Hence, from previous slide, taking $x = 0$, $\max(y(E) : y \leq 0, y \in P_f) = \max(y^-(E) : y \in P_f) = \max(y^-(E) : y \in B_f)$

$\min \{ w^\intercal x : x \in B_f \}$

- Recall that the greedy algorithm solves, for $w \in \mathbb{R}_+^E$
$$\max \{ w^\intercal x | x \in P_f \} = \max \{ w^\intercal x | x \in B_f \} \quad (19.24)$$
since for all $x \in P_f$, there exists $y \geq x$ with $y \in B_f$.
- For arbitrary $w \in \mathbb{R}^E$, greedy algorithm will also solve:
$$\max \{ w^\intercal x | x \in B_f \} \quad (19.25)$$
- Also, since $w \in \mathbb{R}^E$ is arbitrary, and since
$$\min \{ w^\intercal x | x \in B_f \} = -\max \{ -w^\intercal x | x \in B_f \} \quad (19.26)$$
the greedy algorithm using ordering $(e_1, e_2, \dots, e_m)$ such that
$$w(e_1) \leq w(e_2) \leq \dots \leq w(e_m) \quad (19.27)$$
will solve l.h.s. of Equation (19.26).

Greedy solves $\max\{w^\intercal x | x \in B_f\}$ for arbitrary $w \in \mathbb{R}^E$

Let $f(A)$ be arbitrary submodular function, and $f(A) = f'(A) - m(A)$ where f' is polymatroidal, and $w \in \mathbb{R}^E$.

$$\begin{aligned}
\max\{w^\intercal x | x \in B_f\} &= \max\{w^\intercal x | x(A) \le f(A) \, \forall A, x(E) = f(E)\} \\
&= \max\left\{w^\intercal x | x(A) \le f'(A) - m(A) \, \forall A, x(E) = f'(E) - m(E)\right\} \\
&= \max\left\{w^\intercal x | x(A) + m(A) \le f'(A) \, \forall A, x(E) + m(E) = f'(E)\right\} \\
&= \max\Big\{w^\intercal x + w^\intercal m | \\
&\qquad x(A) + m(A) \le f'(A) \, \forall A, x(E) + m(E) = f'(E)\Big\} - w^\intercal m \\
&= \max\left\{w^\intercal y | y \in B_{f'}\right\} - w^\intercal m \\
&= w^\intercal y^* - w^\intercal m = w^\intercal (y^* - m)
\end{aligned}$$

where $y = x + m$, so that $x^* = y^* - m$.
So y^* uses greedy algorithm with positive orthant $B_{f'}$. To show, we use Theorem 11.4.1 in Lecture 11, but we don't require $y \ge 0$, and don't stop when w goes negative to ensure $y^* \in B_{f'}$. Then when we subtract off m from y^*, we get solution to the original problem.

Min-Norm Point and Submodular Function Minimization

- Given optimal solution x^* to $[\min \|x\|_2^2$ s.t. $x \in B_f]$, and consider:

$$y^* = x^* \wedge 0 = (\min(x^*(e), 0) | e \in E), \tag{19.2}$$

$$A_- = \{e : x^*(e) < 0\}, \tag{19.3}$$

$$A_0 = \{e : x^*(e) \le 0\}. \tag{19.4}$$

- Thus, we immediately have that:

$$A_- \subseteq A_0 \tag{19.5}$$

and that

$$x^*(A_-) = x^*(A_0) = y^*(A_-) = y^*(A_0) \tag{19.6}$$

- It turns out, these quantities will solve the submodular function minimization problem, as we now show.
- The proof is nice since it uses the tools we've been recently developing.

Min-Norm Point and SFM

Theorem 19.6.1

Let x^*, y^*, A_-, and A_0 be as given. Then y^* is a maximizer of the l.h.s. of Eqn. (19.7). Moreover, A_- is the unique minimal minimizer of f and A_0 is the unique maximal minimizer of f.

Proof.

- First note, since $x^* \in B_f$, we have $x^*(E) = f(E)$, meaning $\text{sat}(x^*) = E$. Thus, we may consider any $e \in E$ within $\text{dep}(x^*, e)$.
- Consider any pair (e, e') with $e \in A_-$ and $e' \in \text{dep}(x^*, e)$. Then $x^*(e) < 0$, and $\exists \alpha > 0$ s.t. $x^* + \alpha \mathbf{1}_e - \alpha \mathbf{1}_{e'} \in P_f$.
- We have $x^*(E) = f(E)$ and x^* is minimum in l2 sense. We have $(x^* + \alpha \mathbf{1}_e - \alpha \mathbf{1}_{e'}) \in P_f$, and in fact

$$(x^* + \alpha \mathbf{1}_e - \alpha \mathbf{1}_{e'})(E) = x^*(E) + \alpha - \alpha = f(E) \tag{19.28}$$

so $x^* + \alpha \mathbf{1}_e - \alpha \mathbf{1}_{e'} \in B_f$ also.

...

Min-Norm Point and SFM

... proof of Thm. 19.6.1 cont.

- Then $(x^* + \alpha \mathbf{1}_e - \alpha \mathbf{1}_{e'})(E)$ $= x^*(E \setminus \{e, e'\}) + \underbrace{(x^*(e) + \alpha)}_{x^*_{\text{new}}(e)} + \underbrace{(x^*(e') - \alpha)}_{x^*_{\text{new}}(e')} = f(E)$.
- Minimality of $x^* \in B_f$ in l2 sense requires that, with such an $\alpha > 0$, $\Big(x^*(e)\Big)^2 + \Big(x^*(e')\Big)^2 < \Big(x^*_{\text{new}}(e)\Big)^2 + \Big(x^*_{\text{new}}(e')\Big)^2$
- Given that $e \in A_-$, $x^*(e) < 0$. Thus, if $x^*(e') > 0$, we would have $(x^*(e) + \alpha')^2 + (x^*(e') - \alpha')^2 < (x^*(e))^2 + (x^*(e'))^2$, for some $0 < \alpha' \leq \alpha$, contradicting the optimality of x^*.
- If $x^*(e') = 0$, we would have $(x^*(e) + \alpha)^2 + (\alpha')^2 < (x^*(e))^2$, for any $0 < \alpha' < |x^*(e)|$ by convexity, again contradicting optimality of x^*.
- Thus, we must have $x^*(e') < 0$ (strict negativity).

...

Min-Norm Point and SFM

. . . proof of Thm. 19.6.1 cont.

- Thus, for a pair (e, e') with $e' \in \text{dep}(x^*, e)$ and $e \in A_-$, we have $x(e') < 0$ and hence $e' \in A_-$.
- Hence, $\forall e \in A_-$, we have $\text{dep}(x^*, e) \subseteq A_-$.
- A very similar argument can show that, $\forall e \in A_0$, we have $\text{dep}(x^*, e) \subseteq A_0$.
- Also, recall that $e \in \text{dep}(x^*, e)$.

. . .

Min-Norm Point and SFM

. . . proof of Thm. 19.6.1 cont.

- Therefore, we have $\cup_{e \in A_-} \text{dep}(x^*, e) = A_-$ and $\cup_{e \in A_0} \text{dep}(x^*, e) = A_0$
- Ie., $\{\text{dep}(x^*, e)\}_{e \in A_-}$ is cover for A_-, as is $\{\text{dep}(x^*, e)\}_{e \in A_0}$ for A_0.
- $\text{dep}(x^*, e)$ is minimal tight set containing e, meaning $x^*(\text{dep}(x^*, e)) = f(\text{dep}(x^*, e))$, and since tight sets are closed under union, we have that A_- and A_0 are also tight, meaning:
$$x^*(A_-) = f(A_-) \tag{19.29}$$
$$x^*(A_0) = f(A_0) \tag{19.30}$$
$$x^*(A_-) = x^*(A_0) = y^*(E) = y^*(A_0) + \underbrace{y^*(E \setminus A_0)}_{=0} \tag{19.31}$$
and therefore, all together we have
$$f(A_-) = f(A_0) = x^*(A_-) = x^*(A_0) = y^*(E) \tag{19.32}$$
- Hence, $f(A_-) = f(A_0)$, meaning A_- and A_0 have the same valuation, but we have not yet shown they are the minimizers of the submodular function, nor that they are, resp. the maximal and minimal minimizers.

Min-Norm Point and SFM

. . . proof of Thm. 19.6.1 cont.

- Now, y^* is feasible for the l.h.s. of Eqn. (19.7) (recall, which is $\max\{y(E)|y \in P_f, y \leq 0\} = \min\{f(X)|X \subseteq V\}$). This follows since, we have $y^* = x^* \wedge 0 \leq 0$, and since $x^* \in B_f \subset P_f$, and $y^* \leq x^*$ and P_f is down-closed, we have that $y^* \in P_f$.
- Also, for any $y \in P_f$ with $y \leq 0$ and for any $X \subseteq E$, we have $y(E) \leq y(X) \leq f(X)$.
- Hence, we have found a feasible for l.h.s. of Eqn. (19.7), $y^* \leq 0$, $y^* \in P_f$, so $y^*(E) \leq f(X)$ for all X.
- So $y^*(E) \leq \min\{f(X)|X \subseteq V\}$.
- Considering Eqn. (19.33), we have found sets A_- and A_0 with tightness in Eqn. (19.7), meaning $y^*(E) = f(A_-) = f(A_0)$.
- Hence, y^* is a maximizer of l.h.s. of Eqn. (19.7), and A_- and A_0 are minimizers of f.

Min-Norm Point and SFM

. . . proof of Thm. 19.6.1 cont.

- We next show that, not only are they minimizers, but A_- is the unique minimal and A_0 is the unique maximal minimizer of f
- Now, for any $X \subset A_-$, we have

$$f(X) \geq x^*(X) > x^*(A_-) = f(A_-) \tag{19.33}$$

- And for any $X \supset A_0$, we have

$$f(X) \geq x^*(X) > x^*(A_0) = f(A_0) \tag{19.34}$$

- Hence, A_- must be the unique minimal minimizer of f, and A_0 is the unique maximal minimizer of f.

□

Min-Norm Point and SFM

- So, if we have a procedure to compute the min-norm point computation, we can solve SFM.
- Nice thing about previous proof is that it uses both expressions for dep for different purposes.
- This was discovered by Fujishige (in fact the proof above is an expanded version of the one found in the book).
- As we will see, the algorithm (by F. Wolfe) can find this min-norm point, essentially an active-set procedure for quadratic programming. It uses Edmonds's greedy algorithm to make it efficient.
- This is currently the best practical algorithm for general purpose submodular function minimization.
- But its underlying lower-bound strong poly complexity is unknown.

Min-norm point and other minimizers of f

- Recall, that the set of minimizers of f forms a lattice.
- Q: If we take any A with $A_- \subset A \subset A_0$, is A also a minimizer?
- In fact, with x^* the min-norm point, and A_- and A_0 as defined above, we have the following theorem:

Theorem 19.6.2

Let $A \subseteq E$ be any minimizer of submodular f, and let x^* be the minimum-norm point. Then A can be expressed in the form:

$$A = A_- \cup \bigcup_{a \in A_m} \text{dep}(x^*, a) \tag{19.35}$$

for some set $A_m \subseteq A_0 \setminus A_-$. Conversely, for any set $A_m \subseteq A_0 \setminus A_-$, then $A \triangleq A_- \cup \bigcup_{a \in A_m} \text{dep}(x^*, a)$ is a minimizer.

Min-norm point and other minimizers of f

proof of Thm. 19.6.2.

- If A is a minimizer, then $A_- \subseteq A \subseteq A_0$, and $f(A) = y^*(E)$ is the minimum valuation of f.
- But $x^* \in P_f$, so $x^*(A) \leq f(A)$ and $f(A) = x^*(A_-) \leq x^*(A)$.
- Also, since $A \subseteq A_0$ and $x^*(A_0 \setminus A) = 0$, $x^*(A_-) = x^*(A) = x^*(A_0)$
- Hence, $x^*(A) = x^*(A_-) = f(A)$ so that A is also a tight set for x^*.
- For any $a \in A$, A is a tight set containing a, and $\text{dep}(x^*, a)$ is the minimal tight containing a.
- Hence, for any $a \in A$, $\text{dep}(x^*, a) \subseteq A$.
- This means that $\bigcup_{a \in A} \text{dep}(x^*, a) = A$.
- Since $A_- \subseteq A \subseteq A_0$, then $\exists A_m \subseteq A \setminus A_-$ such that

$$A = \bigcup_{a \in A_-} \text{dep}(x^*, a) \cup \bigcup_{a \in A_m} \text{dep}(x^*, a) = A_- \cup \bigcup_{a \in A_m} \text{dep}(x^*, a)$$

...

Min-norm point and other minimizers of f

proof of Thm. 19.6.2.

- Conversely, consider any set $A_m \subseteq A_0 \setminus A_-$, and define A as

$$A = A_- \cup \bigcup_{a \in A_m} \text{dep}(x^*, a) = \bigcup_{a \in A_-} \text{dep}(x^*, a) \cup \bigcup_{a \in A_m} \text{dep}(x^*, a) \tag{19.36}$$

- Then since A is a union of tight sets, A is also a tight set, and we have $f(A) = x^*(A)$.
- But $x^*(A \setminus A_-) = 0$, so $f(A) = x^*(A) = x^*(A_-) = f(A_-)$ meaning A is also a minimizer of f.

□

Therefore, we can generate the entire lattice of minimizers of f starting from A_- and A_0 given access to $\text{dep}(x^*, e)$.

On a unique minimizer f

- Note that if $f(e|A) > 0, \forall A \subseteq E$ and $e \in E \setminus A$, then we have $A_- = A_0$ (there is one unique minimizer).
- On the other hand, if $A_- = A_0$, it does not imply $f(e|A) > 0$ for all $A \subseteq E \setminus \{e\}$.
- If $A_- = A_0$ then certainly $f(e|A_0) > 0$ for $e \in E \setminus A_0$ and $-f(e|A_0 \setminus \{e\}) > 0$ for all $e \in A_0$.

Duality: convex minimization of L.E. and min-norm alg.

- Let f be a submodular function with $\tilde{f}$ it's Lovász extension. Then the following two problems are duals (Bach-2013):

$$\underset{w \in \mathbb{R}^V}{\text{minimize}} \ \tilde{f}(w) + \frac{1}{2}\|w\|_2^2 \qquad (19.37)$$

$$\begin{aligned} &\text{maximize} && -\|x\|_2^2 && (19.38a) \\ &\text{subject to} && x \in B_f && (19.38b) \end{aligned}$$

  where $B_f = P_f \cap \left\{x \in \mathbb{R}^V : x(V) = f(V)\right\}$ is the base polytope of submodular function f, and $\|x\|_2^2 = \sum_{e \in V} x(e)^2$ is squared 2-norm.
- Equation (19.37) is related to proximal methods to minimize the Lovász extension (see Parikh&Boyd, "Proximal Algorithms" 2013).
- Equation (19.38b) is solved by the minimum-norm point algorithm (Wolfe-1976, Fujishige-1984, Fujishige-2005, Fujishige-2011) is (as we will see) essentially an active-set procedure for quadratic programming, and uses Edmonds's greedy algorithm to make it efficient.
- Unknown strongly poly worst-case running time, although in practice it usually performs quite well (see below).

Convex and affine hulls, affinely independent

- Given points set $P = \{p_1, p_2, \dots, p_k\}$ with $p_i \in \mathbb{R}^V$, let $\text{conv}\, P$ be the convex hull of P, i.e.,

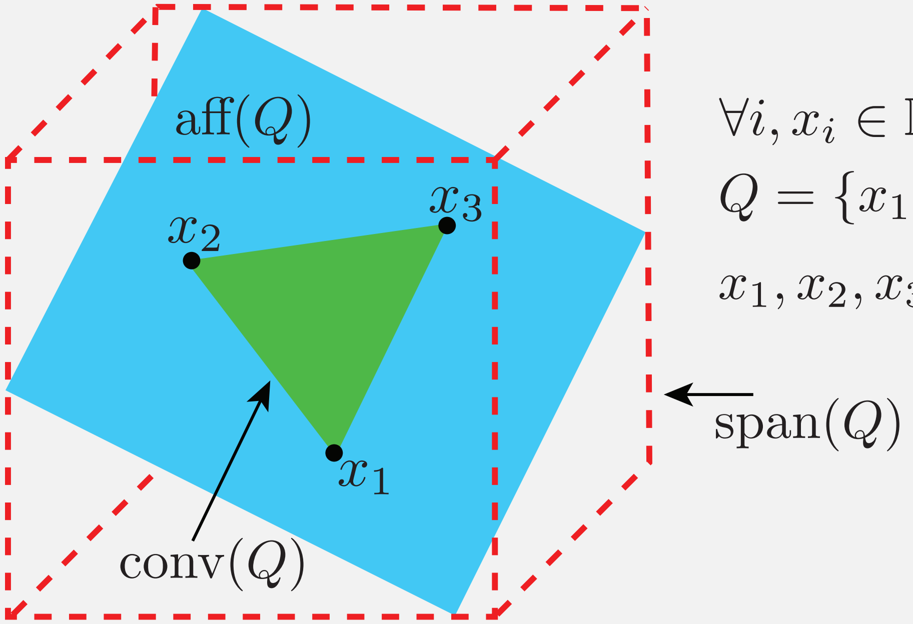
$$\text{conv}\, P \triangleq \left\{ \sum_{i=1}^{k} \lambda_i p_i : \sum_i \lambda_i = 1,\ \lambda_i \geq 0, i \in [k] \right\}. \qquad (19.39)$$

- For a set of points $Q = \{q_1, q_2, \dots, q_k\}$, with $q_i \in \mathbb{R}^V$, we define $\text{aff}\, Q$ to be the affine hull of Q, i.e.:

$$\text{aff}\, Q \triangleq \left\{ \sum_{i \in 1}^{k} \lambda_i q_i : \sum_{i=1}^{k} \lambda_i = 1 \right\} \supseteq \text{conv}\, Q. \qquad (19.40)$$

- A set of points Q is affinely independent if no point in Q belows to the affine hull of the remaining points.

Convex vs. Affine hull, geometry



$\forall i, x_i \in \mathbb{R}^3$

$Q = \{x_1, x_2, x_3\}$

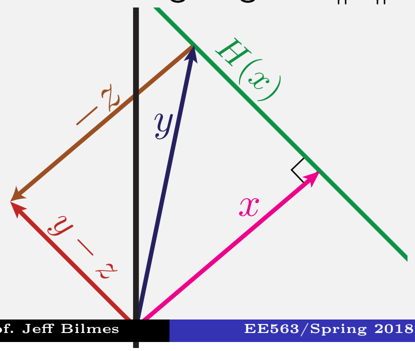
x_1, x_2, x_3 coplanar

$H(x)$: Orthogonal x-containing hyperplane

- Define $H(x)$ as the hyperplane that is orthogonal to the line from 0 to x, while also containing x, i.e.

$$H(x) \triangleq \left\{y \in \mathbb{R}^V \,|\, x^\intercal y = \|x\|_2^2\right\} \tag{19.41}$$

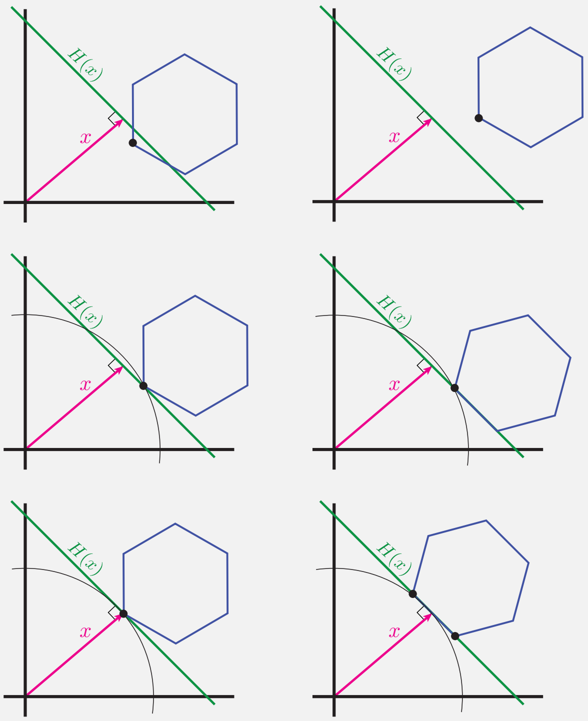
- Any set $\{y \in \mathbb{R}^V | x^\intercal y = c\}$ is orthogonal to the line from 0 to x. This follows since, for constant z, $\{y : (y - z)^\intercal x = 0\} = \{y : y^\intercal x = z^\intercal x\}$ is hyperplane orthogonal to x translated by z. Take $c = z^\intercal x$ for result, and $z = x$, giving $c = \|x\|^2$, to contain x.



- Note, $H(x)$ is translation of subspace of dimension $|V| - 1 = n - 1$ (i.e., $H(x) - \{x\}$ is a subspace, $H(x)$ is an affine set).

Ex: $H(x)$, polytopes, and supporting hyperplanes

- $H(x) = \left\{y \in \mathbb{R}^V | x^\intercal y = \|x\|_2^2\right\}$, any $z \in H(x)$ has $x^\intercal z = x^\intercal x$.
- Consider $\operatorname{conv} P$ polytope for points $P = \{p_1, p_2, \dots\}$, and $\hat{p} \in \operatorname{argmin}_{p \in P} x^\intercal p$. TL: $x^\intercal p < x^\intercal x$; TR: $x^\intercal p > x^\intercal x$; middle row: $x^\intercal p = x^\intercal x$.
- Bottom Row: In Algo, x is chosen so that if $x^\intercal \hat{p} = x^\intercal x$ then $H(x)$ separates P from the origin, and x is the min 2-norm point. Notice that $x^\intercal p \geq x^\intercal x$ for all $p \in P$.
- Middle/bottom row: $H(x)$ is a supporting hyperplane of $\operatorname{conv} P$ (contained, touching).



Notation

- The line between x and y: given two points $x, y \in \mathbb{R}^V$, let $[x, y] \triangleq \{\lambda x + (1 - \lambda y) : \lambda \in [0, 1]\}$. Hence, $[x, y] = \text{conv}\,\{x, y\}$.
- Note, if we wish to minimize the 2-norm of a vector $\|x\|_2$, we can equivalently minimize its square $\|x\|_2^2 = \sum_i x_i^2$, and vice verse.

Frank-Wolfe vs. Fujishige-Wolfe

An algorithm we will <u>not</u> use for the min-norm is M. Frank & P. Wolfe “An algorithm for quadratic programming”, 1956, or conditional gradient descent for constrained convex minimization given convex function $f : \mathcal{D} \to \mathbb{R}$.

```
  Input  : Convex $f : \mathcal{D} \to \mathbb{R}$, $x_0 \in \mathcal{D}$
  Output: $x^* \in \mathcal{D}$, the minimizer of $f$.
1 $k \leftarrow 0$ and start with $x_0 \in \mathcal{D}$ ;
2 Let $s_k$ solve $\min\langle s, \nabla f(x_k)\rangle$ s.t. $s \in \mathcal{D}$ ;
3 Let $\lambda_k \in [0, 1]$ minimize $f(\lambda s_k + (1 - \lambda)x_k)$ ;
4 $x_{k+1} \leftarrow \lambda_k s_k + (1 - \lambda_k)x_k$, $k \leftarrow k + 1$ ;
5 Goto line 1 if $\|x_{k+1} - x_k\| > \tau$ ;
6 $x^* \leftarrow x_{k+1}$
```

- Above could minimize Lovász extension, primal approach to SFM.
- For finding the min-norm point, we will be using the P. Wolfe, “Finding the Nearest Point in a Polytope”, 1976 which is the same Wolfe but different algorithm and different year.

Fujishige-Wolfe Min-Norm Algorithm

- Wolfe-1976 ("Finding the Nearest Point in a Polytope") developed an algorithm to compute the minimum norm point of a polytope, specified as a set of vertices (again, not same as Frank-Wolfe'1956).
- Given set of points $P = \{p_1, \cdots, p_m\}$ where $p_i \in \mathbb{R}^n$: find the minimum norm point in convex hull of P:

$$\min_{x \in \text{conv } P} \|x\|_2 \tag{19.42}$$

- Wolfe's algorithm is guaranteed terminating, and explicitly uses a representation of x as a convex combination of points in P
- Fujishige-1984 "Submodular Systems and Related Topics" realized this algorithm can find the the min. norm point of B_f thanks to Edmond's greedy algorithm.
- Seems to still be (among) the fastest general purpose SFM algo.
- Algorithm maintains a set of points $Q \subseteq P$, which is always assuredly *affinely independent*.

Fujishige-Wolfe Min-Norm Algorithm

- When Q are affinely independent, minimum norm point in the affine hull of Q can easily be found, as a closed form solution for $\min_{x \in \text{aff } Q} \|x\|_2$ is available (see below).
- Algorithm repeatedly produces min. norm point x^* for selected set Q.
- If we find $w_i \geq 0, i = 1, \cdots, m$ for the minimum norm point, then x^* also belongs to $\text{conv } Q$ and also a minimum norm point over $\text{conv } Q$.
- If $Q \subseteq P$ is suitably chosen, x^* may even be the minimum norm point over $\text{conv } P$ solving the original problem.
- One of the most expensive parts of Wolfe's original 1976 algorithm is solving linear optimization problem over the polytope, doable by examining all the extreme points in the polytope.
- If number of extreme points is exponential, hard to do in general.
- Number of extreme points of submodular base polytope is exponentially large, but linear optimization over the base polytope B_f doable $O(n \log n)$ time via Edmonds's greedy algorithm.

Pseudocode of Fujishige-Wolfe Min-Norm (MN) algorithm

```
   Input  : $P = \{p_1, \cdots, p_m\}, p_i \in \mathbb{R}^n, i = 1, \cdots, m.$
   Output: $x^*$: the minimum-norm-point in $\operatorname{conv} P$.
1  $x^* \longleftarrow p_{i^*}$ where $p_{i^*} \in \operatorname{argmin}_{p \in P} \|p\|_2$    /* or choose it arbitrarily */ ;
2  $Q \longleftarrow \{x^*\}$;
3  while 1 do                                                  /* major loop */
4      if $x^* = 0$ or $H(x^*)$ separates $P$ from origin then
           return : $x^*$
5      else
6          Choose $\hat{x} \in P$ on the near (closer to $0$) side of $H(x^*)$;
7          $Q = Q \cup \{\hat{x}\}$;
8      while 1 do                                              /* minor loop */
9          $x_0 \longleftarrow \operatorname{argmin}_{x \in \operatorname{aff} Q} \|x\|_2$;
10         if $x_0 \in \operatorname{conv} Q$ then
11             $x^* \longleftarrow x_0$;
12             break;
13         else
14             $y \longleftarrow \operatorname{argmin}_{x \in \operatorname{conv} Q \cap [x^*, x_0]} \|x - x_0\|_2$;
15             Delete from $Q$ points not on the face of $\operatorname{conv} Q$ where $y$ lies;
16             $x^* \longleftarrow y$;
```

Fujishige-Wolfe Min-Norm algorithm: Geometric Example

- It is advised that for the next set of slides, you have a print out of the previous MN algorithm available on display/paper somewhere.
- Algorithm maintains an invariant, namely that:

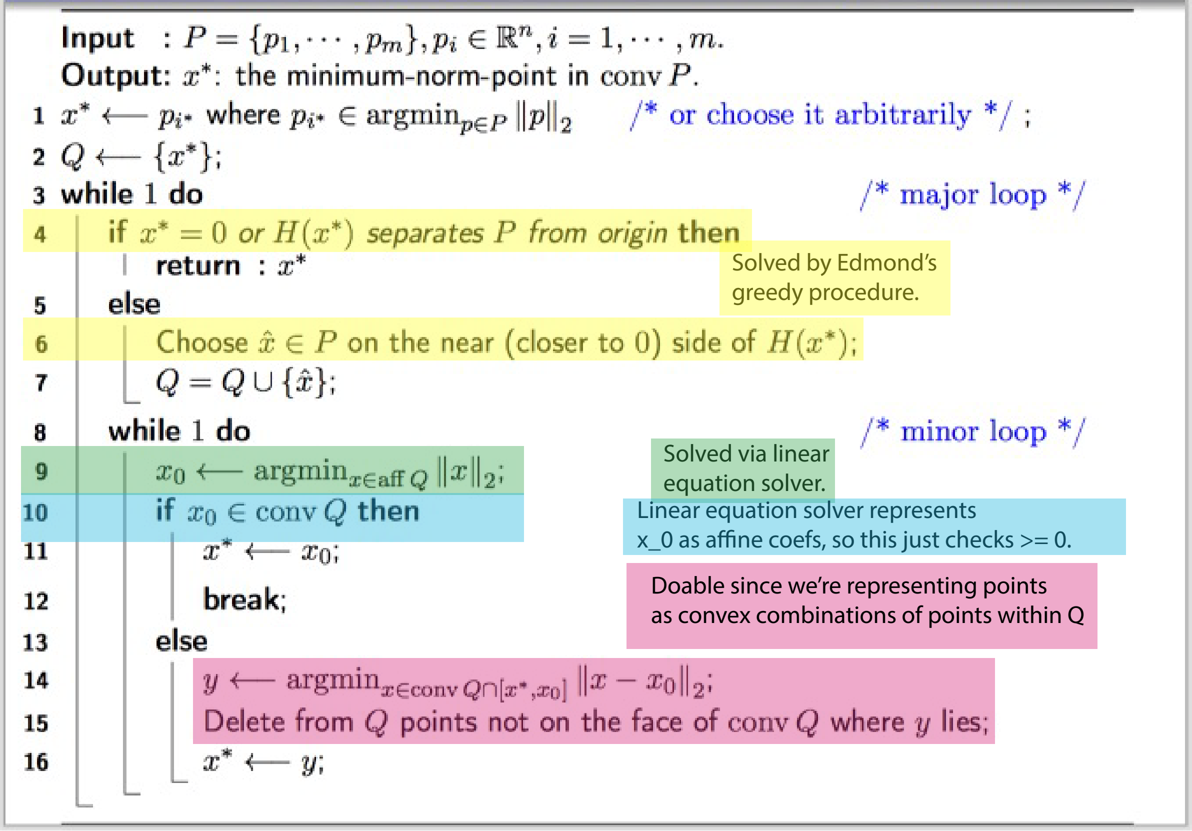
$$x^* \in \operatorname{conv} Q \subseteq \operatorname{conv} P, \tag{19.43}$$

  must hold at every possible assignment of x^* (Lines 1, 11, and 16):
  1. True after Line 1 since $Q = \{x^*\}$,
  2. True after Line 11 since $x_0 \in \operatorname{conv} Q$,
  3. and true after Line 16 since $y \in \operatorname{conv} Q$ even after deleting points.
- Note also for any $x^* \in \operatorname{conv} Q \subseteq \operatorname{conv} P$, we have

$$\min_{x \in \operatorname{aff} Q} \|x\|_2 \leq \min_{x \in \operatorname{conv} Q} \|x\|_2 \leq \|x^*\|_2 \tag{19.44}$$

- Note, the input, P, consists of m points. In the case of the base polytope, $P = B_f$ could be exponential in $n = |V|$.
- There are six places that might be seemingly tricky or expensive: Line 4, Line 6, Line 9, Line 10, Line 14, and Line 15.
- We will consider each in turn, but first we do a geometric example.

Pseudocode of Fujishige-Wolfe Min-Norm (MN) algorithm



```
Input  : $P = \{p_1, \cdots, p_m\}, p_i \in \mathbb{R}^n, i = 1, \cdots, m$.
Output: $x^*$: the minimum-norm-point in conv $P$.
$x^* \longleftarrow p_{i^*}$ where $p_{i^*} \in \mathrm{argmin}_{p \in P} \|p\|_2$    /* or choose it arbitrarily */ ;
$Q \longleftarrow \{x^*\}$;
while 1 do                                                  /* major loop */
    if $x^* = 0$ or $H(x^*)$ separates $P$ from origin then
        return : $x^*$
    else
        Choose $\hat{x} \in P$ on the near (closer to 0) side of $H(x^*)$;
        $Q = Q \cup \{\hat{x}\}$;
    while 1 do                                              /* minor loop */
        $x_0 \longleftarrow \mathrm{argmin}_{x \in \mathrm{aff}\, Q} \|x\|_2$;
        if $x_0 \in \mathrm{conv}\, Q$ then
            $x^* \longleftarrow x_0$;
            break;
        else
            $y \longleftarrow \mathrm{argmin}_{x \in \mathrm{conv}\, Q \cap [x^*, x_0]} \|x - x_0\|_2$;
            Delete from $Q$ points not on the face of conv $Q$ where $y$ lies;
            $x^* \longleftarrow y$;
```

Solved by Edmond's greedy procedure.

Solved via linear equation solver.

Linear equation solver represents x_0 as affine coefs, so this just checks >= 0.

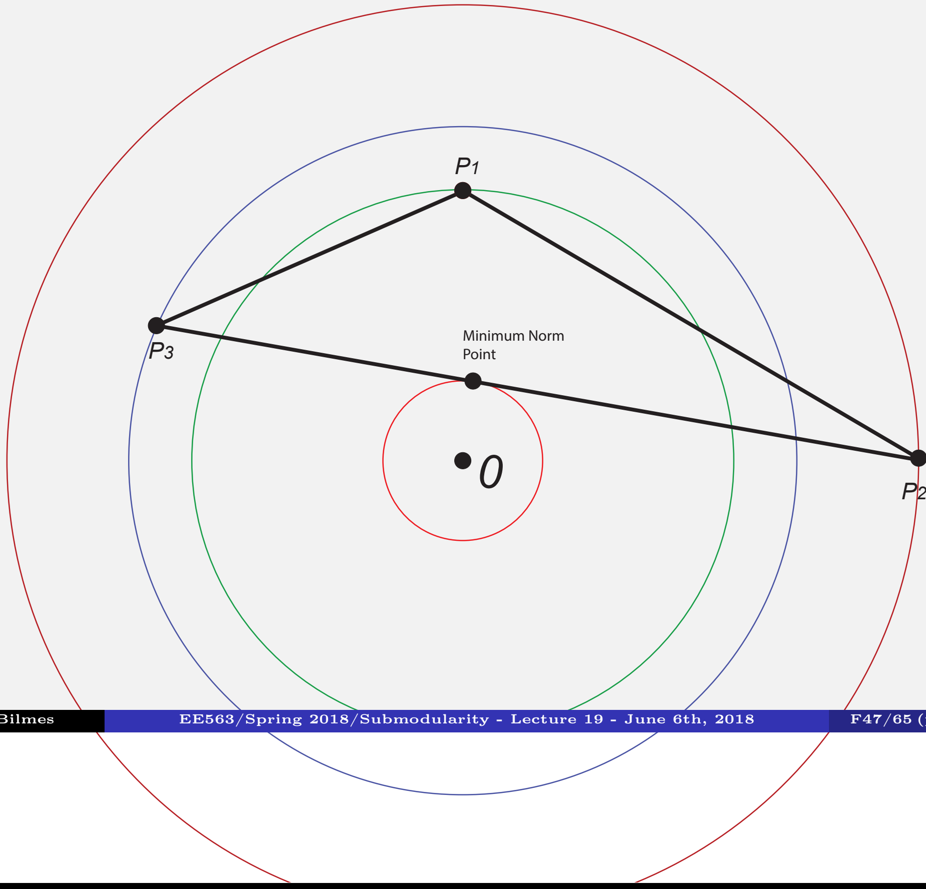
Doable since we're representing points as convex combinations of points within Q

Fujishige-Wolfe Min-Norm algorithm: Geometric Example

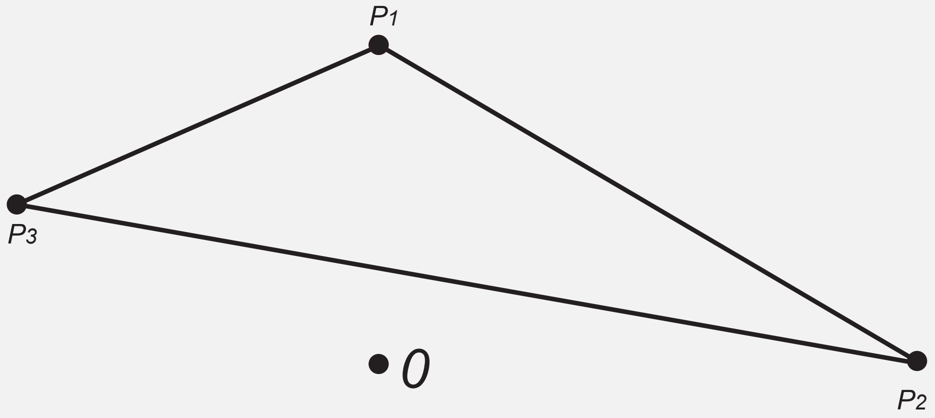
- In the following series of images, permanent (non-changing) named points on the polytope will be indicated by capital letters (i.e., P_1, P_2, P_3, R, S, T) while variables in the algorithm that are changing will use lower case letters (i.e., x^*, x_0, $\hat{x}$, y).
- Also, example is in 2D, so polytope given can't be a real base B_f for any f. Example meant to show only the geometry of the algorithm.

Fujishige-Wolfe Min-Norm algorithm: Geometric Example

Polytope, and circles concentric at 0.

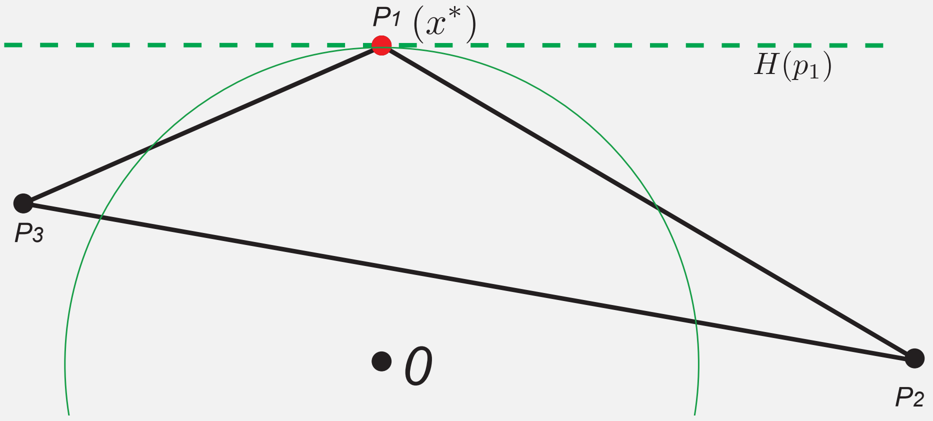


Fujishige-Wolfe Min-Norm algorithm: Geometric Example



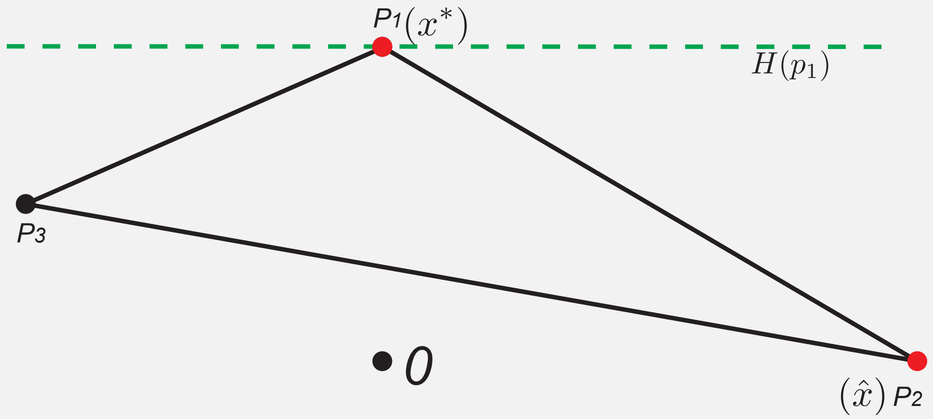
The initial polytope consisting of the convex hull of three points p_1, p_2, p_3, and the origin 0.

Fujishige-Wolfe Min-Norm algorithm: Geometric Example



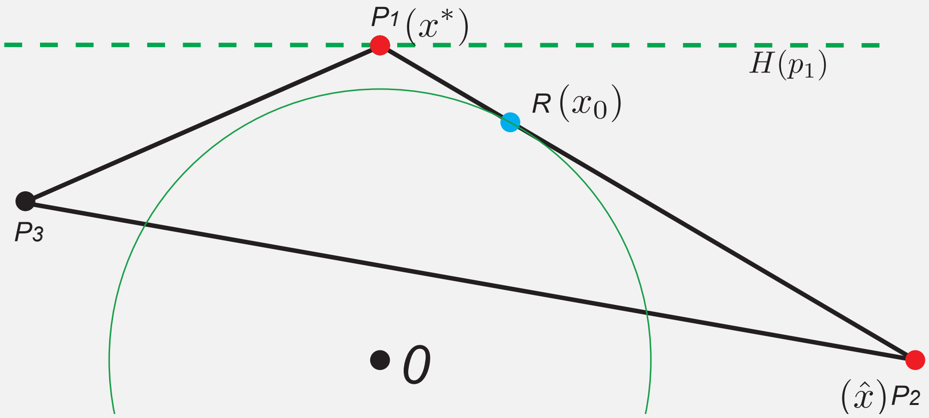
p_1 is the extreme point closest to 0 and so we choose it first, although we can choose any arbitrary extreme point as the initial point. We set $x^* \leftarrow p_1$ in Line 1, and $Q \leftarrow \{p_1\}$ in Line 2. $H(x^*) = H(p_1)$ (green dashed line) is not a supporting hyperplane of $\text{conv}(P)$ in Line 4, so we move on to the else condition in Line 5.

Fujishige-Wolfe Min-Norm algorithm: Geometric Example



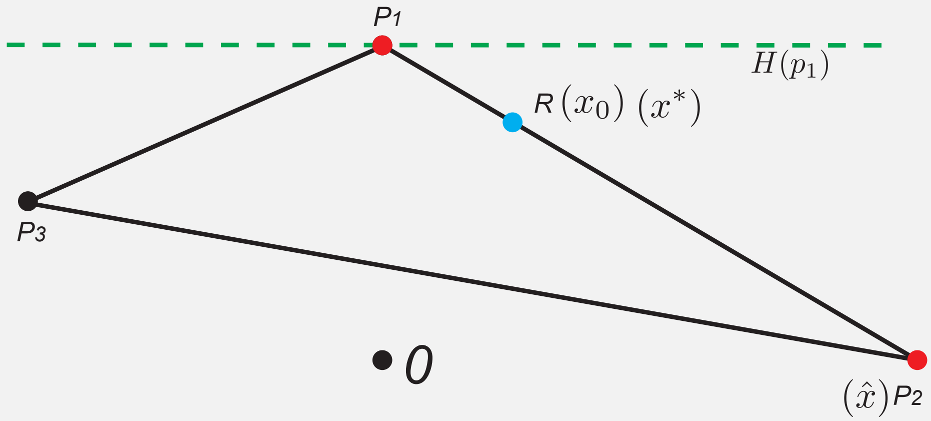
We need to add some extreme point $\hat{x}$ on the "near" side of $H(p_1)$ in Line 6, we choose $\hat{x} = p_2$. In Line 7, we set $Q \leftarrow Q \cup \{p_2\}$, so $Q = \{p_1, p_2\}$.

Fujishige-Wolfe Min-Norm algorithm: Geometric Example



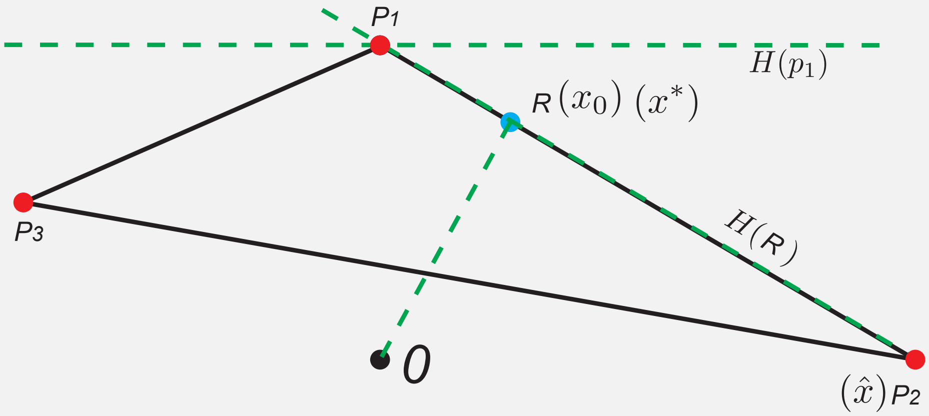
$x_0 = R$ is the min-norm point in $\operatorname{aff}\{p_1, p_2\}$ computed in Line 9. Also, with $Q = \{p_1, p_2\}$, since $R \in \operatorname{conv} Q$, we set $x^* \leftarrow x_0 = R$ in Line 11, not violating the invariant $x^* \in \operatorname{conv} Q$. Note, after Line 11, we still have $x^* \in \operatorname{conv} P$ and $\|x^*\|_2 = \|x^*_{\text{new}}\|_2 < \|x^*_{\text{old}}\|_2$ strictly.

Fujishige-Wolfe Min-Norm algorithm: Geometric Example



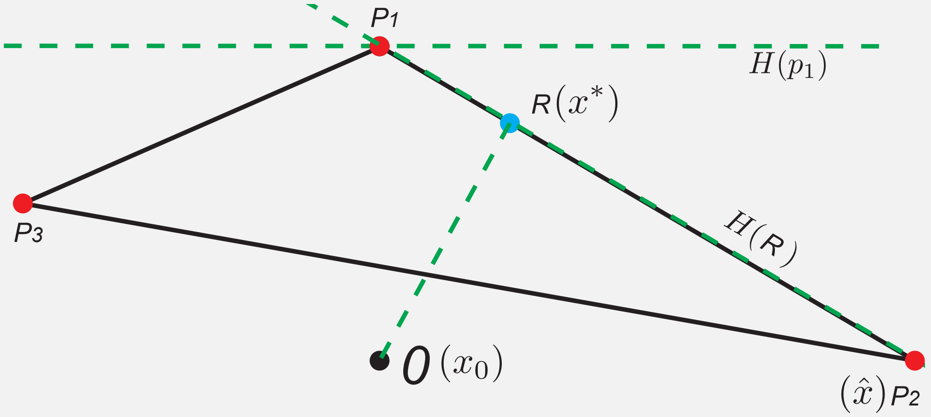
$x_0 = R$ is the min-norm point in $\operatorname{aff}\{p_1, p_2\}$ computed in Line 9. Also, with $Q = \{p_1, p_2\}$, since $R \in \operatorname{conv} Q$, we set $x^* \leftarrow x_0 = R$ in Line 11, not violating the invariant $x^* \in \operatorname{conv} Q$. Note, after Line 11, we still have $x^* \in \operatorname{conv} P$ and $\|x^*\|_2 = \|x^*_{\text{new}}\|_2 < \|x^*_{\text{old}}\|_2$ strictly.

Fujishige-Wolfe Min-Norm algorithm: Geometric Example



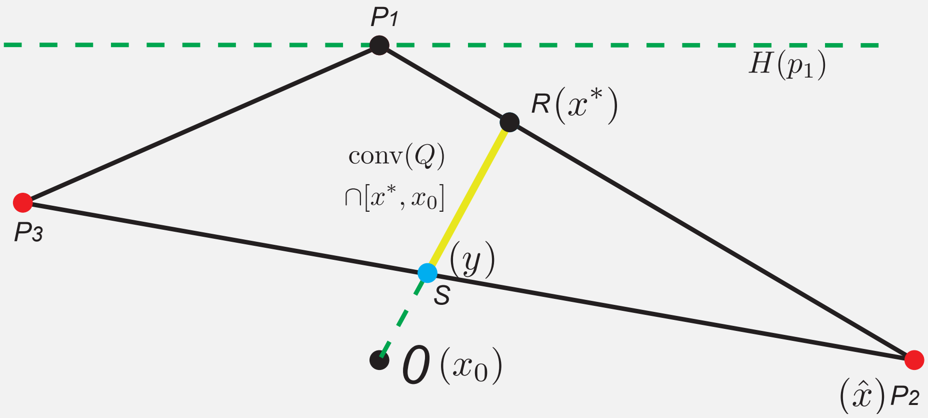
$R = x_0 = x^*$. We consider next $H(R) = H(x^*)$ in Line 4. $H(x^*)$ is not a supporting hyperplane of $\operatorname{conv} P$. So we choose p_3 on the "near" side of $H(x^*)$ in Line 6. Add $Q \leftarrow Q \cup \{p_3\}$ in Line 7. Now $Q = P = \{p_1, p_2, p_3\}$. The origin $x_0 = 0$ is the min-norm point in $\operatorname{aff} Q$ (Line 9), and it is not in the interior of $\operatorname{conv} Q$ (condition in Line 10 is false).

Fujishige-Wolfe Min-Norm algorithm: Geometric Example



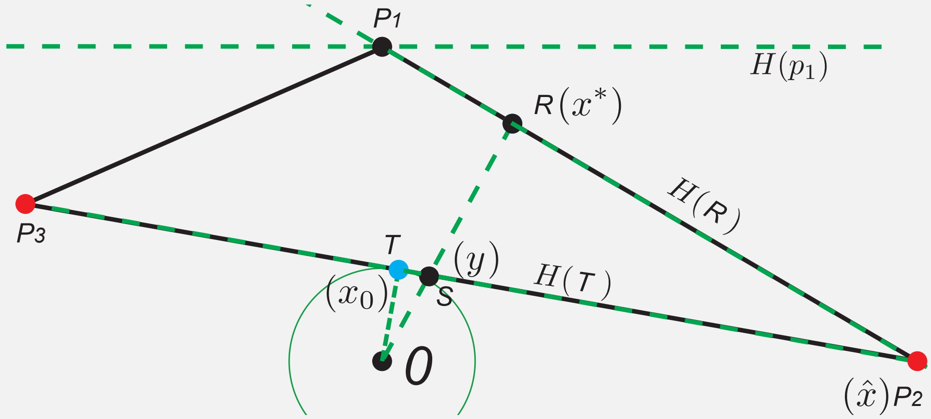
$R = x_0 = x^*$. We consider next $H(R) = H(x^*)$ in Line 4. $H(x^*)$ is not a supporting hyperplane of $\operatorname{conv} P$. So we choose p_3 on the "near" side of $H(x^*)$ in Line 6. Add $Q \leftarrow Q \cup \{p_3\}$ in Line 7. Now $Q = P = \{p_1, p_2, p_3\}$. The origin $x_0 = 0$ is the min-norm point in $\operatorname{aff} Q$ (Line 9), and it is not in the interior of $\operatorname{conv} Q$ (condition in Line 10 is false).

Fujishige-Wolfe Min-Norm algorithm: Geometric Example



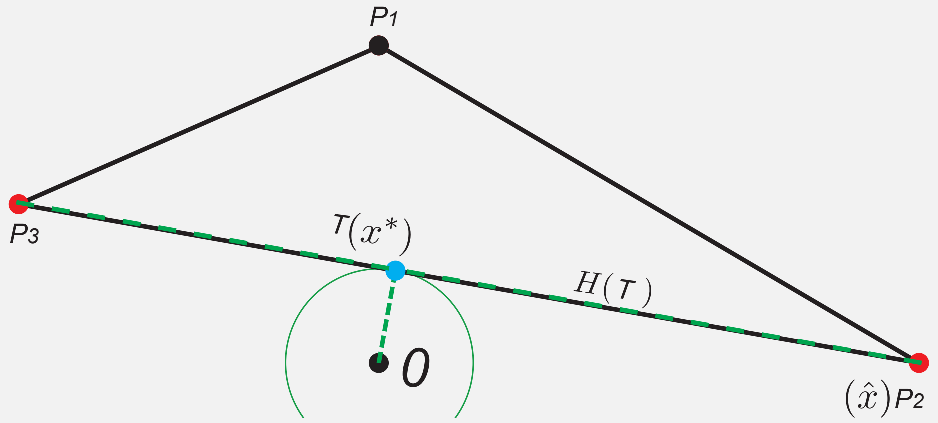
$Q = P = \{p_1, p_2, p_3\}$. Line 14: $S = y = \text{argmin}_{x \in \text{conv}\, Q \cap [x^*, x_0]} \|x - x_0\|_2$ where x_0 is 0 and x^* is R here. Thus, y lies on the boundary of $\text{conv}\, Q$. Note, $\|y\|_2 < \|x^*\|_2$ since $x^* \in \text{conv}\, Q$, $\|x_0\|_2 < \|x^*\|_2$. Line 15: Delete p_1 from Q since not on face where $y = S$ lies. $Q = \{p_2, p_3\}$ after Line 15. We still have $y = S \in \text{conv}\, Q$ for the updated Q. Line 16: $x^* \leftarrow y$, retain invariant $x^* \in \text{conv}\, Q$, and again have $\|x^*\|_2 = \|x^*_{\text{new}}\|_2 < \left\|x^*_{\text{old}}\right\|_2$ strictly.

Fujishige-Wolfe Min-Norm algorithm: Geometric Example



$Q = \{p_2, p_3\}$, and so $x_0 = T$ computed in Line 9 is the min-norm point in $\text{aff}\, Q$. We also have $x_0 \in \text{conv}\, Q$ in Line 10 so we assign $x^* \leftarrow x_0$ in Line 11 and break.

Fujishige-Wolfe Min-Norm algorithm: Geometric Example



$H(T)$ separates P from the origin in Line 4, and therefore is a supporting hyperplane, and therefore x^* is the min-norm point in $\operatorname{conv} P$, so we return with x^*.

Condition for Min-Norm Point

Theorem 19.7.1

$P = \{p_1, p_2, \dots, p_m\}$, $x^* \in \operatorname{conv} P$ *is the min. norm point in* $\operatorname{conv} P$ *iff*

$$p_i^\intercal x^* \geq \|x^*\|_2^2 \quad \forall i = 1, \cdots, m. \tag{19.45}$$

Proof.

- Assume x^* is the min-norm point, let $y \in \operatorname{conv} P$, and $0 \leq \theta \leq 1$.
- Then $z \triangleq x^* + \theta(y - x^*) = (1-\theta)x^* + \theta y \in \operatorname{conv} P$, and

$$\|z\|_2^2 = \|x^* + \theta(y - x^*)\|_2^2 \tag{19.46}$$

$$= \|x^*\|_2^2 + 2\theta(x^{*\intercal}y - x^{*\intercal}x^*) + \theta^2 \|y - x^*\|_2^2 \tag{19.47}$$

- It is possible for $\|z\|_2^2 < \|x^*\|_2^2$ for small θ, unless $x^{*\intercal}y \geq x^{*\intercal}x^*$ for all $y \in \operatorname{conv} P \Rightarrow$ Equation (19.45).
- Conversely, given Eq (19.45), and given that $y = \sum_i \lambda_i p_i \in \operatorname{conv} P$,

$$y^\intercal x^* = \sum_i \lambda_i p_i^\intercal x^* \geq \sum_i \lambda_i x^{*\intercal} x^* = x^{*\intercal} x^* \tag{19.48}$$

implying that $\|z\|_2^2 > \|x^*\|_2^2$ in Equation 19.47 for arbitrary $z \in \operatorname{conv} P$. □

The set Q is always affinely independent

Lemma 19.7.2

The set Q in the MN Algorithm is always affinely independent.

Proof.

- Q is of course affinely independent when there is at most one point in it (e.g., after Line 2).
- After the initialization, it changes only by deletion of points, or adding a single point. Deletion does not change the independence.
- Before adding $\hat{x}$ at Line 7, we know x^* is the minimum norm point in $\operatorname{aff} Q$ (since we break only at Line 12).
- Therefore, x^* is normal to $\operatorname{aff} Q$, which implies $\operatorname{aff} Q \subseteq H(x^*)$.
- Since $\hat{x} \notin H(x^*)$ chosen at Line 6, we have $\hat{x} \notin \operatorname{aff} Q$.
- $\therefore$ update $Q \cup \{\hat{x}\}$ at Line 7 is affinely independent as long as Q is. □

Thus, by Lemma 19.7.2, we have for any $x \in \operatorname{aff} Q$ such that $x = \sum_i w_i q_i$ with $\sum_i w_i = 1$, the weights w_i are uniquely determined.

The set Q is never too large

Lemma 19.7.3

The set Q in the MN Algorithm has size never more than $n + 1$.

Proof.

This is immediate, since Q is always affinely independnet, and in $\mathbb{R}^V$, an affinely independnet set can have at most $n + 1$ entries, with $|V| = n$. □

Minimum Norm in an affine set

- Line 9 of the algorithm requires $x_0 \leftarrow \min_{x \in \text{aff } Q} \|x\|_2$.
- When Q is affinely independent, this is relatively easy.
- Let Q represent $n \times k$ matrix with points as columns $q \in Q$. The following is solvable with matrix inversion/linear solver, where $x = Qw$:

$$\text{minimize} \qquad \|x\|_2^2 = w^\intercal Q^\intercal Q w \tag{19.49}$$

$$\text{subject to} \qquad \mathbf{1}^\intercal w = 1 \tag{19.50}$$

- Form Lagrangian $w^\intercal Q^\intercal Q w + 2\lambda(\mathbf{1}^\intercal w - 1)$, and differentiating w.r.t. λ and w, and setting to zero, we get:

$$\mathbf{1}^\intercal w = 1 \tag{19.51}$$

$$Q^\intercal Q w + \lambda \mathbf{1} = 0 \tag{19.52}$$

- $k+1$ variables and k unknowns, solvable with linear solver with matrices

$$\begin{bmatrix} 0 & \mathbf{1}^\intercal \\ \mathbf{1} & Q^\intercal Q \end{bmatrix} \begin{bmatrix} \lambda \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix} \tag{19.53}$$

- Thanks to Q being affine, matrix on l.h.s. is invertable.

Minimum Norm in an affine set

- Note, this also solves Line 10, since feasibility requires $\sum_i w_i = 1$, we need only check $w \geq 0$ to ensure $x_0 = \sum_i w_i q_i \in \text{conv } Q$.
- In fact, a feature of the algorithm (in Wolfe's 1976 paper) is that we keep the convex coefficients $\{w_i\}_i$ where $x^* = \sum_i w_i p_i$ of x^* and from this vector. We also keep v such that $x_0 = \sum_i v_i q_i$ for points $q_i \in Q$, from Line 9.
- Given w and v, we can also easily solve Lines 14 and 15 (see "Step 3" on page 133 of Wolfe-1976, which also defines numerical tolerances).
- We have yet to see how to efficiently solve Lines 4 and 6, however.

MN Algorithm finds the MN point in finite time.

Theorem 19.7.4

The MN Algorithm finds the minimum norm point in $\operatorname{conv} P$ *after a finite number of iterations of the major loop.*

Proof.

- In minor loop, we always have $x^* \in \operatorname{conv} Q$, since whenever Q is modified, x^* is updated as well (Line 16) such that the updated x^* remains in new $\operatorname{conv} Q$.
- Hence, every time x^* is updated (in minor loop), its norm never increases, i.e., before Line 11, $\|x_0\|_2 \leq \|x^*\|_2$ since $x^* \in \operatorname{aff} Q$ and $x_0 = \min_{x \in \operatorname{aff} Q} \|x\|_2$. Similarly, before Line 16, $\|y\|_2 \leq \|x^*\|_2$, since invariant $x^* \in \operatorname{conv} Q$ but while $x_0 \in \operatorname{aff} Q$, we have $x_0 \notin \operatorname{conv} Q$, and $\|x_0\|_2 < \|x^*\|_2$.

. . .

MN Algorithm finds the MN point in finite time.

. . . proof of Theorem 19.7.4 continued.

- Moreover, there can be no more iterations within a minor loop than the dimension of $\operatorname{conv} Q$ for the initial Q given to the minor loop initially at Line 8 (dimension of $\operatorname{conv} Q$ is $|Q| - 1$ since Q is affinely independent).
- Each iteration of the minor loop removes at least one point from Q in Line 15.
- When Q reduces to a singleton, the minor loop always terminates.
- Thus, the minor loop terminates in finite number of iterations, at most dimension of Q.
- In fact, total number of iterations of minor loop in entire algorithm is at most number of points in P since we never add back in points to Q that have been removed.

. . .

MN Algorithm finds the MN point in finite time.

. . . proof of Theorem 19.7.4 continued.

- Each time Q is augmented with $\hat{x}$ at Line 7, followed by updating x^* with x_0 at Line 11, (i.e., when the minor loop returns with only one iteration), $\|x^*\|_2$ strictly decreases from what it was before.
- To see this, consider $x^* + \theta(\hat{x} - x^*)$ where $0 \leq \theta \leq 1$. Since both $\hat{x}, x^* \in \operatorname{conv} Q$, we have $x^* + \theta(\hat{x} - x^*) \in \operatorname{conv} Q$.
- Therefore, we have $\|x^* + \theta(\hat{x} - x^*)\|_2 \geq \|x_0\|_2$, which implies

$$\begin{aligned} \|x^* + \theta(\hat{x} - x^*)\|_2^2 &= \|x^*\|_2^2 + 2\theta\left((x^*)^\top \hat{x} - \|x^*\|_2^2\right) + \theta^2 \|\hat{x} - x^*\|_2^2 \\ &\geq \|x_0\|_2^2 \end{aligned} \tag{19.54}$$

and from Line 6, $\hat{x}$ is on the same side of $H(x^*)$ as the origin, i.e. $(x^*)^\top \hat{x} < \|x^*\|_2^2$, so middle term of r.h.s. of equality is negative.

. . .

MN Algorithm finds the MN point in finite time.

. . . proof of Theorem 19.7.4 continued.

- Therefore, for sufficiently small θ, specifically for

$$\theta < \frac{2\left(\|x^*\|_2^2 - (x^*)^\top \hat{x}\right)}{\|\hat{x} - x^*\|_2^2} \tag{19.55}$$

we have that $\|x^*\|_2^2 > \|x_0\|_2^2$.

- For a similar reason, we have $\|x^*\|_2$ strictly decreases each time Q is updated at Line 7 and followed by updating x^* with y at Line 16.
- Therefore, in each iteration of major loop, $\|x^*\|_2$ strictly decreases, and the MN Algorithm must terminate and it can only do so when the optimal is found.

□

Line: 6: Finding $\hat{x} \in P$ on the near side of $H(x^*)$

- The "near" side means the side that contains the origin.
- Ideally, find $\hat{x}$ such that the reduction of $\|x^*\|_2$ is maximized to reduce number of major iterations.
- From Eqn. 19.54, reduction on norm is lower-bounded:

$$\Delta = \|x^*\|_2^2 - \|x_0\|_2^2 \geq 2\theta\left(\|x^*\|_2^2 - (x^*)^\top \hat{x}\right) - \theta^2 \|\hat{x} - x^*\|_2^2 \triangleq \underline{\Delta} \tag{19.56}$$

- When $0 \leq \theta < \frac{2\left(\|x^*\|_2^2 - (x^*)^\top \hat{x}\right)}{\|\hat{x} - x^*\|_2^2}$, we can get the maximal value of the lower bound, over θ, as follows:

$$\max_{0 \leq \theta < \frac{2\left(\|x^*\|_2^2 - (x^*)^\top \hat{x}\right)}{\|\hat{x} - x^*\|_2^2}} \underline{\Delta} = \left(\frac{\|x^*\|_2^2 - (x^*)^\top \hat{x}}{\|\hat{x} - x^*\|_2}\right)^2 \tag{19.57}$$

Line: 6: Finding $\hat{x} \in P$ on the near side of $H(x^*)$

- To maximize lower bound of norm reduction at each major iteration, want to find an $\hat{x}$ such that the above lower bound (Equation 19.57) is maximized.
- That is, we want to find

$$\hat{x} \in \underset{x \in P}{\operatorname{argmax}} \left(\frac{\|x^*\|_2^2 - (x^*)^\top x}{\|x - x^*\|_2}\right)^2 \tag{19.58}$$

  to ensure that a large norm reduction is assured.
- This problem, however, is at least as hard as the MN problem itself as we have a quadratic term in the denominator.

Line: 6: Finding $\hat{x} \in P$ on the near side of $H(x^*)$

- As a surrogate, we maximize numerator in Eqn. 19.58, i.e., find

$$\hat{x} \in \underset{x \in P}{\operatorname{argmax}} \|x^*\|_2^2 - (x^*)^\top x = \underset{x \in P}{\operatorname{argmin}} (x^*)^\top x, \tag{19.59}$$

- Intuitively, by solving the above, we find $\hat{x}$ such that it has the largest "distance" to the hyperplane $H(x^*)$, and this is exactly the strategy used in the Wolfe-1976 algorithm.
- Also, solution $\hat{x}$ in Line 6 can be used to determine if hyperplane $H(x^*)$ separates $\operatorname{conv} P$ from the origin (Line 4): if the point in P having greatest distance to $H(x^*)$ is not on the side where origin lies, then $H(x^*)$ separates $\operatorname{conv} P$ from the origin.
- Mathematically and theoretically, we terminate the algorithm if

$$(x^*)^\top \hat{x} \geq \|x^*\|_2^2, \tag{19.60}$$

where $\hat{x}$ is the solution of Eq. 19.59.

Line: 6: Finding $\hat{x} \in P$ on the near side of $H(x^*)$

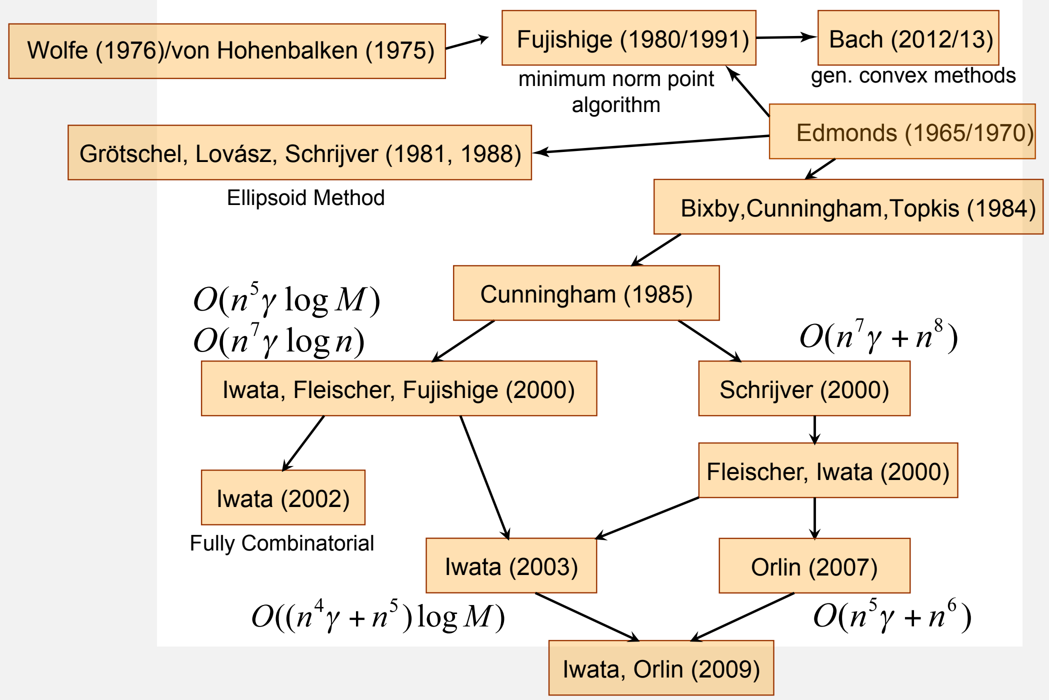
- In practice,the above optimality test might never hold numerically. Hence, as suggested by Wolfe, we introduce a tolerance parameter $\epsilon > 0$, and terminates the algorithm if

$$(x^*)^\top \hat{x} > \|x^*\|_2^2 - \epsilon \max_{x \in Q} \|x\|_2^2 \tag{19.61}$$

- When $\operatorname{conv} P$ is a submodular base polytope (i.e., $\operatorname{conv} P = B_f$ for a submodular function f), then the problem in Eqn 19.59 can be solved efficiently by Edmonds's greedy algorithm (even though there may be an exponential number of extreme points).
- Edmond's greedy algorithm, therefore, solves both Line 4 and Line 6 simultaneously.
- Hence, Edmonds's discovery is one of the main reasons that the MN algorithm is applicable to submodular function minimization.

SFM Summary (modified from S. Iwata's slides)

General Submodular Function Minimization



Wolfe (1976)/von Hohenbalken (1975)

Fujishige (1980/1991)

minimum norm point algorithm

Bach (2012/13)

gen. convex methods

Edmonds (1965/1970)

Grötschel, Lovász, Schrijver (1981, 1988)

Ellipsoid Method

Bixby,Cunningham,Topkis (1984)

Cunningham (1985)

$O(n^5\gamma \log M)$
$O(n^7\gamma \log n)$

Iwata, Fleischer, Fujishige (2000)

$O(n^7\gamma + n^8)$

Schrijver (2000)

Fleischer, Iwata (2000)

Iwata (2002)

Fully Combinatorial

Iwata (2003)

Orlin (2007)

$O((n^4\gamma + n^5)\log M)$

$O(n^5\gamma + n^6)$

Iwata, Orlin (2009)

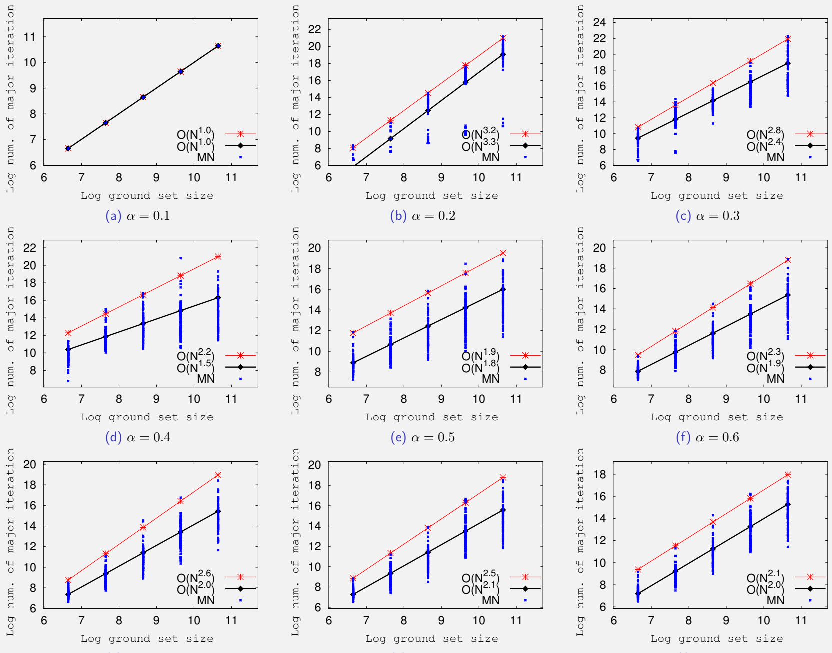
MN Algorithm Complexity

- The currently fastest strongly polynomial combinatorial algorithm for SFM achieves a running time of $O(n^5T + n^6)$ (Orlin'09) where T is the time for function evaluation, far from practical for large problem instances.
- Fujishige & Isotani report that MN algorithm is fast in practice, but they use only a limited set of submodular functions.
- Complexity of MN Algorithm is still an unsolved problem.
- Obvious facts:
  - each major iteration requires $O(n)$ function oracle calls
  - complexity of each major iteration could be at least $O(n^3)$ due to the affine projection step (solving a linear system).
  - Therefore, the complexity of each major iteration is

    $$O(n^3 + n^{1+p})$$

    where each function oracle call requires $O(n^p)$ time.
- Since the number of major iterations required is unknown, the complexity of MN is also unknown.

MN Algorithm Empirical Complexity



(a) $\alpha = 0.1$ (b) $\alpha = 0.2$ (c) $\alpha = 0.3$

(d) $\alpha = 0.4$ (e) $\alpha = 0.5$ (f) $\alpha = 0.6$

(g) $\alpha = 0.7$ (h) $\alpha = 0.8$ (i) $\alpha = 0.9$

Figure: The number of major iteration for $f(S) = -m_1(S) + 100 \cdot (w_1(\mathcal{N}(S)))^\alpha$. The red lines are the linear interpolations of the worst case points, and the black lines are the linear interpolations of the average case points. From Lin&Bilmes 2014 (unpublished)

MN Algorithm Complexity

- A lower bound complexity of the min-norm has not been established.
- In 2014, Chakrabarty, Jain, and Kothari in their NIPS 2014 paper "Provable Submodular Minimization using Wolfe's Algorithm" showed a pseudo-polynomial time bound of $O(n^7 g_f^2)$ where $n = |V|$ is the ground set, and g_f is the maximum gain of a particular function f.
- This is pseudo-polynomial since it depends on the function values.
- There currently is no known polynomial time complexity analysis for this algorithm.