Submodular Functions, Optimization, and Applications to Machine Learning
— Spring Quarter, Lecture 6 —

http://www.ee.washington.edu/people/faculty/bilmes/classes/ee563_spring_2018/

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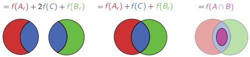
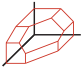
April 11th, 2018

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$





Cumulative Outstanding Reading

- Read chapter 1 from Fujishige's book.
- Read chapter 2 from Fujishige's book.

Announcements, Assignments, and Reminders

- If you have any questions about anything, please ask then via our discussion board (https://canvas.uw.edu/courses/1216339/discussion_topics).

Class Road Map - EE563

- L1(3/26): Motivation, Applications, & Basic Definitions,
- L2(3/28): Machine Learning Apps (diversity, complexity, parameter, learning target, surrogate).
- L3(4/2): Info theory exs, more apps, definitions, graph/combinatorial examples
- L4(4/4): Graph and Combinatorial Examples, Matrix Rank, Examples and Properties, visualizations
- L5(4/9): More Examples/Properties/ Other Submodular Defs., Independence,
- L6(4/11): Matroids, Matroid Examples, Matroid Rank, Partition/Laminar Matroids
- L7(4/16): Laminar Matroids, System of Distinct Reps, Transversals, Transversal Matroid, Matroid Representation, Dual Matroids
- L8(4/18):
- L9(4/23):
- L10(4/25):
- L11(4/30):
- L12(5/2):
- L13(5/7):
- L14(5/9):
- L15(5/14):
- L16(5/16):
- L17(5/21):
- L18(5/23):
- L–(5/28): Memorial Day (holiday)
- L19(5/30):
- L21(6/4): Final Presentations maximization.

Last day of instruction, June 1st. Finals Week: June 2-8, 2018.

Composition of non-decreasing submodular and non-decreasing concave

Theorem 6.2.1

Given two functions, one defined on sets

$$f : 2^V \to \mathbb{R} \tag{6.1}$$

and another continuous valued one:

$$\phi : \mathbb{R} \to \mathbb{R} \tag{6.2}$$

the composition formed as $h = \phi \circ f : 2^V \to \mathbb{R}$ (defined as $h(S) = \phi(f(S))$) is nondecreasing submodular, if ϕ is non-decreasing concave and f is nondecreasing submodular.

Monotone difference of two functions

Let f and g both be submodular functions on subsets of V and let $(f-g)(\cdot)$ be either monotone non-decreasing or monotone non-increasing Then $h : 2^V \to R$ defined by

$$h(A) = \min(f(A), g(A)) \tag{6.1}$$

is submodular.

Proof.

If $h(A)$ agrees with f on both X and Y (or g on both X and Y), and since

$$h(X) + h(Y) = f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y) \tag{6.2}$$

or

$$h(X) + h(Y) = g(X) + g(Y) \geq g(X \cup Y) + g(X \cap Y), \tag{6.3}$$

the result (Equation ?? being submodular) follows since

$$\begin{matrix} f(X) + f(Y) \\ g(X) + g(Y) \end{matrix} \geq \min(f(X \cup Y), g(X \cup Y)) + \min(f(X \cap Y), g(X \cap Y)) \tag{6.4}$$

...

Arbitrary functions: difference between submodular funcs.

Theorem 6.2.1

Given an arbitrary set function h, it can be expressed as a difference between two submodular functions (i.e., $\forall h \in 2^V \to \mathbb{R}$, $\exists f, g$ s.t. $\forall A, h(A) = f(A) - g(A)$ where both f and g are submodular).

Proof.

Let h be given and arbitrary, and define:

$$\alpha \triangleq \min_{X,Y : X \not\subseteq Y, Y \not\subseteq X} \Big(h(X) + h(Y) - h(X \cup Y) - h(X \cap Y) \Big) \tag{6.4}$$

If $\alpha \geq 0$ then h is submodular, so by assumption $\alpha < 0$. Now let f be an arbitrary strict submodular function and define

$$\beta \triangleq \min_{X,Y : X \not\subseteq Y, Y \not\subseteq X} \Big(f(X) + f(Y) - f(X \cup Y) - f(X \cap Y) \Big). \tag{6.5}$$

Strict means that $\beta > 0$. . . .

Many (Equivalent) Definitions of Submodularity

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B), \;\; \forall A, B \subseteq V \tag{6.16}$$

$$f(j|S) \geq f(j|T), \; \forall S \subseteq T \subseteq V, \text{ with } j \in V \setminus T \tag{6.17}$$

$$f(C|S) \geq f(C|T), \forall S \subseteq T \subseteq V, \text{ with } C \subseteq V \setminus T \tag{6.18}$$

$$f(j|S) \geq f(j|S \cup \{k\}), \; \forall S \subseteq V \text{ with } j \in V \setminus (S \cup \{k\}) \tag{6.19}$$

$$f(A \cup B|A \cap B) \leq f(A|A \cap B) + f(B|A \cap B), \;\; \forall A, B \subseteq V \tag{6.20}$$

$$f(T) \leq f(S) + \sum_{j \in T \setminus S} f(j|S) - \sum_{j \in S \setminus T} f(j|S \cup T - \{j\}), \; \forall S, T \subseteq V \tag{6.21}$$

$$f(T) \leq f(S) + \sum_{j \in T \setminus S} f(j|S), \; \forall S \subseteq T \subseteq V \tag{6.22}$$

$$f(T) \leq f(S) - \sum_{j \in S \setminus T} f(j|S \setminus \{j\}) + \sum_{j \in T \setminus S} f(j|S \cap T) \; \forall S, T \subseteq V \tag{6.23}$$

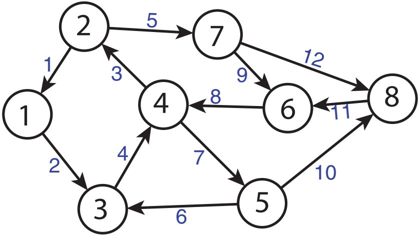
$$f(T) \leq f(S) - \sum_{j \in S \setminus T} f(j|S \setminus \{j\}), \; \forall T \subseteq S \subseteq V \tag{6.24}$$

On Rank

- Let rank : $2^V \to \mathbb{Z}_+$ be the rank function.
- In general, $\text{rank}(A) \leq |A|$, and vectors in A are linearly independent if and only if $\text{rank}(A) = |A|$.
- If A, B are such that $\text{rank}(A) = |A|$ and $\text{rank}(B) = |B|$, with $|A| < |B|$, then the space spanned by B is greater, and we can find a vector in B that is linearly independent of the space spanned by vectors in A.
- To stress this point, note that the above condition is $|A| < |B|$, **not** $A \subseteq B$ which is sufficient (to be able to find an independent vector) but not required.
- In other words, given A, B with $\text{rank}(A) = |A|$ & $\text{rank}(B) = |B|$, then $|A| < |B| \Leftrightarrow \exists$ an $b \in B$ such that $\text{rank}(A \cup \{b\}) = |A| + 1$.

Spanning trees/forests & incidence matrices

- A directed version of the graph (right) and its adjacency matrix (below).
- Orientation can be arbitrary.
- Note, rank of this matrix is 7.



	1	2	3	4	5	6	7	8	9	10	11	12
1	−1	1	0	0	0	0	0	0	0	0	0	0
2	1	0	−1	0	1	0	0	0	0	0	0	0
3	0	−1	0	1	0	−1	0	0	0	0	0	0
4	0	0	1	−1	0	0	1	−1	0	0	0	0
5	0	0	0	0	0	1	−1	0	0	1	0	0
6	0	0	0	0	0	0	0	1	−1	0	−1	0
7	0	0	0	0	−1	0	0	0	1	0	0	1
8	0	0	0	0	0	0	0	0	0	−1	1	−1

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- maxInd: Inclusionwise maximal independent subsets (i.e., the set of bases of) of any set $B \subseteq V$ defined as:

$$\text{maxInd}(B) \triangleq \{A \subseteq B : A \in \mathcal{I} \text{ and } \forall v \in B \setminus A, A \cup \{v\} \notin \mathcal{I}\} \tag{6.2}$$

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- Given any set $B \subset V$ of vectors, all maximal (by set inclusion) subsets of linearly independent vectors are the same size. That is, for all $B \subseteq V$,

$$\forall A_1, A_2 \in \text{maxInd}(B), \quad |A_1| = |A_2| = \text{rank}(B) \tag{6.3}$$

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- Let $\mathcal{I} = \{I_1, I_2, \ldots\}$ be the set of sets as described above.
- Thus, for all $I \in \mathcal{I}$, the matrix rank function has the property

$$r(I) = |I| \tag{6.4}$$

and for any $B \notin \mathcal{I}$,

$$r(B) = \max\{|A| : A \subseteq B \text{ and } A \in \mathcal{I}\} < |B| \tag{6.5}$$

Since all maximally independent subsets of a set are the same size, the rank function is well defined.

Matroid

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- In a matroid, there is an underlying ground set, say E (or V), and a collection of subsets $\mathcal{I} = \{I_1, I_2, \ldots\}$ of E that correspond to independent elements.
- There are many definitions of matroids that are mathematically equivalent, we'll see some of them here.

Independence System

Definition 6.3.1 (set system)

A (finite) ground set E and a set of subsets of E, $\emptyset \neq \mathcal{I} \subseteq 2^E$ is called a set system, notated $(E, \mathcal{I})$.

- Set systems can be arbitrarily complex since, as stated, there is no systematic method (besides exponential-cost exhaustive search) to determine if a given set $S \subseteq E$ has $S \in \mathcal{I}$.

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- Set systems can be arbitrarily complex since, as stated, there is no systematic method (besides exponential-cost exhaustive search) to determine if a given set $S \subseteq E$ has $S \in \mathcal{I}$.
- One useful property is "heredity." Namely, a set system is a hereditary set system if for any $A \subset B \in \mathcal{I}$, we have that $A \in \mathcal{I}$.

Independence System

Definition 6.3.2 (independence (or hereditary) system)

A set system $(V, \mathcal{I})$ is an independence system if

$$\emptyset \in \mathcal{I} \quad \text{(emptyset containing)} \tag{I1}$$

and

$$\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I} \quad \text{(subclusive)} \tag{I2}$$

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- Then $(E, \mathcal{I})$ is a set system, but not an independence system since it is not down closed (e.g., we have $\{1, 2\} \in \mathcal{I}$ but not $\{2\} \in \mathcal{I}$).
- With $\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, then $(E, \mathcal{I})$ is now an independence (hereditary) system.

Independence System

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 0 & 1 & 1 & 2 & 1 & 3 & 1 \\ 0 & 1 & 1 & 0 & 2 & 0 & 2 & 4 \\ 1 & 1 & 1 & 0 & 0 & 3 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ | & | & | & | & | & | & | & | \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ | & | & | & | & | & | & | & | \end{pmatrix} \quad (6.6)$$

- Given any set of linearly independent vectors A, any subset $B \subset A$ will also be linearly independent.

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- Given any set of linearly independent vectors A, any subset $B \subset A$ will also be linearly independent.
- Given any forest G_f that is an edge-induced sub-graph of a graph G, any sub-graph of G_f is also a forest.

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- Given any set of linearly independent vectors A, any subset $B \subset A$ will also be linearly independent.
- Given any forest G_f that is an edge-induced sub-graph of a graph G, any sub-graph of G_f is also a forest.
- So these both constitute independence systems.

Matroid

Independent set definition of a matroid is perhaps most natural. Note, if $J \in \mathcal{I}$, then J is said to be an independent set.

Definition 6.3.3 (Matroid)

A set system $(E, \mathcal{I})$ is a Matroid if

(I1) $\emptyset \in \mathcal{I}$

(I2) $\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}$ (down-closed or subclusive)

(I3) $\forall I, J \in \mathcal{I}$, with $|I| = |J| + 1$, then there exists $x \in I \setminus J$ such that $J \cup \{x\} \in \mathcal{I}$.

Why is (I1) is not redundant given (I2)?

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Why is (I1) is not redundant given (I2)? Because without (I1) could have a non-matroid where $\mathcal{I} = \{\}$.

On Matroids

- Abstract properties of linear dependence (Hassler Whitney, 1935), but already then found instances of objects with those properties not based on a matrix.

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- Crapo & Rota preferred the term "combinatorial geometry", or more specifically a "pregeometry" and said that pregeometries are "often described by the ineffably cacaphonic [sic] term 'matroid', which we prefer to avoid in favor of the term 'pregeometry'."

Matroid

Slight modification (non unit increment) that is equivalent.

Definition 6.3.4 (Matroid-II)

A set system $(E, \mathcal{I})$ is a Matroid if

(I1') $\emptyset \in \mathcal{I}$

(I2') $\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}$ (down-closed or subclusive)

(I3') $\forall I, J \in \mathcal{I}$, with $|I| > |J|$, then there exists $x \in I \setminus J$ such that $J \cup \{x\} \in \mathcal{I}$

Note (I1)=(I1'), (I2)=(I2'), and we get (I3)≡(I3') using induction.

Matroids, independent sets, and bases

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- A base of a matroid: If $U = E$, then a "base of E" is just called a base of the matroid M (this corresponds to a basis in a linear space, or a spanning forest in a graph, or a spanning tree in a connected graph).

Matroids - important property

Proposition 6.3.5

In a matroid $M = (E, \mathcal{I})$, for any $U \subseteq E(M)$, any two bases of U have the same size.

Matroids - important property

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(I3') $\forall X \subseteq V$, and $I_1, I_2 \in \text{maxInd}(X)$, we have $|I_1| = |I_2|$ (all maximally independent subsets of X have the same size).

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- From the above, we immediately see that $r(A) \leq |A|$.
- Moreover, if $r(A) = |A|$, then $A \in \mathcal{I}$, meaning A is independent (in this case, A is a self base).

Matroids, other definitions using matroid rank $r : 2^V \to \mathbb{Z}_+$

Definition 6.3.8 (closed/flat/subspace)

A subset $A \subseteq E$ is closed (equivalently, a flat or a subspace) of matroid M if for all $x \in E \setminus A$, $r(A \cup \{x\}) = r(A) + 1$.

Definition: A hyperplane is a flat of rank $r(M) - 1$.

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Definition 6.3.10 (circuit)

A subset $A \subseteq E$ is circuit or a cycle if it is an inclusionwise-minimal dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).

Matroids by bases

In general, besides independent sets and rank functions, there are other equivalent ways to characterize matroids.

Theorem 6.3.11 (Matroid (by bases))

Let E be a set and $\mathcal{B}$ be a nonempty collection of subsets of E. Then the following are equivalent.

1. *$\mathcal{B}$ is the collection of bases of a matroid;*
2. *if $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B' - x + y \in \mathcal{B}$ for some $y \in B \setminus B'$.*
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Properties 2 and 3 are called "exchange properties."
Proof here is omitted but think about this for a moment in terms of linear spaces and matrices, and (alternatively) spanning trees.

Matroids by circuits

A set is independent if and only if it contains no circuit. Therefore, it is not surprising that circuits can also characterize a matroid.

Theorem 6.3.12 (Matroid by circuits)

Let E be a set and $\mathcal{C}$ be a collection of subsets of E that satisfy the following three properties:

1. *(C1):* $\emptyset \notin \mathcal{C}$
2. *(C2): if* $C_1, C_2 \in \mathcal{C}$ *and* $C_1 \subseteq C_2$, *then* $C_1 = C_2$.
3. *(C3): if* $C_1, C_2 \in \mathcal{C}$ *with* $C_1 \neq C_2$, *and* $e \in C_1 \cap C_2$, *then there exists a* $C_3 \in \mathcal{C}$ *such that* $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$.

Matroids by circuits

Several circuit definitions for matroids.

Theorem 6.3.13 (Matroid by circuits)

Let E be a set and $\mathcal{C}$ be a collection of nonempty subsets of E, such that no two sets in $\mathcal{C}$ are contained in each other. Then the following are equivalent.

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Again, think about this for a moment in terms of linear spaces and matrices, and spanning trees.

Uniform Matroid

- Given E, consider $\mathcal{I}$ to be all subsets of E that are at most size k. That is $\mathcal{I} = \{A \subseteq E : |A| \leq k\}$.

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- Note, if $I, J \in \mathcal{I}$, and $|I| < |J| \leq k$, and $j \in J$ such that $j \notin I$, then j is such that $|I + j| \leq k$ and so $I + j \in \mathcal{I}$.

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- A "free" matroid sets $k = |E|$, so everything is independent.

Linear (or Matric) Matroid

- Let $\mathbf{X}$ be an $n \times m$ matrix and $E = \{1, \dots, m\}$
- Let $\mathcal{I}$ consists of subsets of E such that if $A \in \mathcal{I}$, and $A = \{a_1, a_2, \dots, a_k\}$ then the vectors $x_{a_1}, x_{a_2}, \dots, x_{a_k}$ are linearly independent.
- the rank function is just the rank of the space spanned by the corresponding set of vectors.
- rank is submodular, it is intuitive that it satisfies the diminishing returns property (a given vector can only become linearly dependent in a greater context, thereby no longer contributing to rank).
- Called both linear matroids and matric matroids.

Cycle Matroid of a graph: Graphic Matroids

- Let $G = (V, E)$ be a graph. Consider $(E, \mathcal{I})$ where the edges of the graph E are the ground set and $A \in \mathcal{I}$ if the edge-induced graph $G(V, A)$ by A does not contain any cycle.

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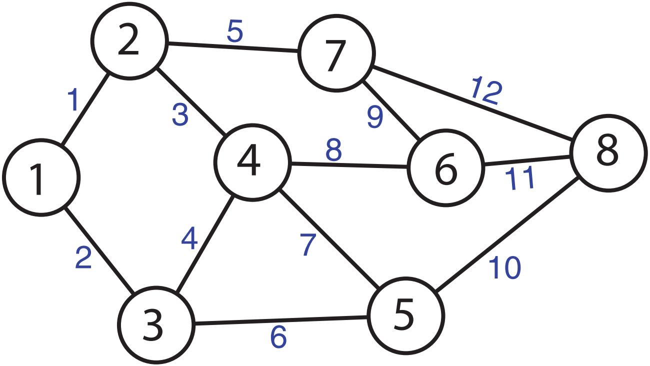
- Closure function adds all edges between the vertices adjacent to any edge in A. Closure of a spanning forest is G.

Example: graphic matroid

- A graph defines a matroid on edge sets, independent sets are those without a cycle.



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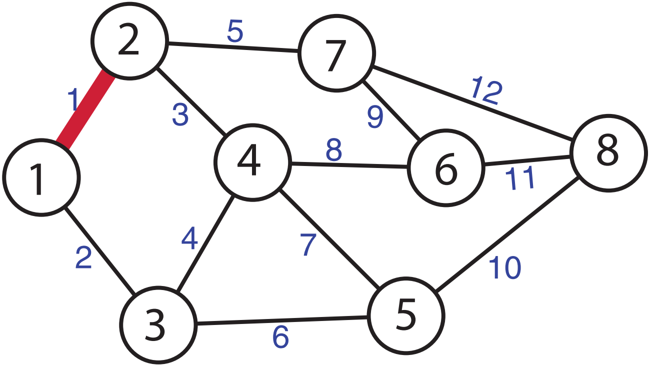

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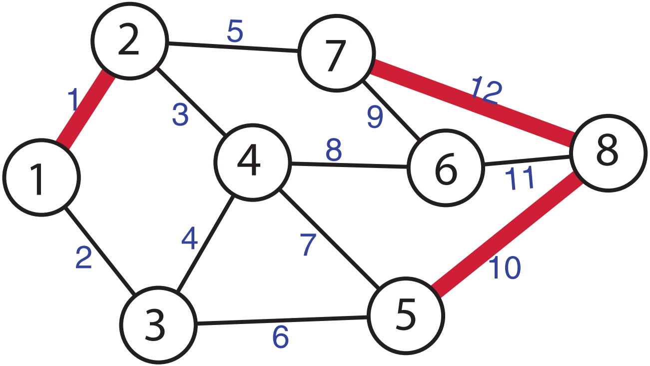

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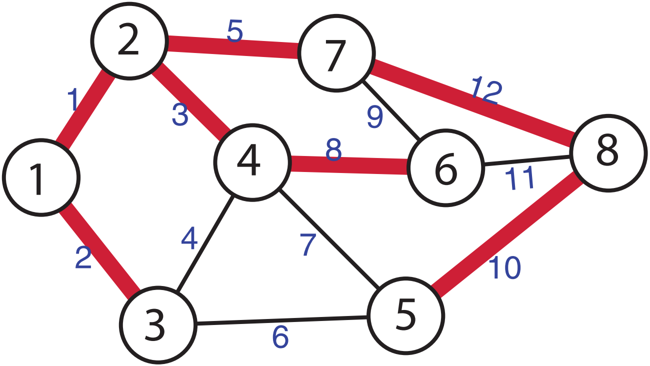

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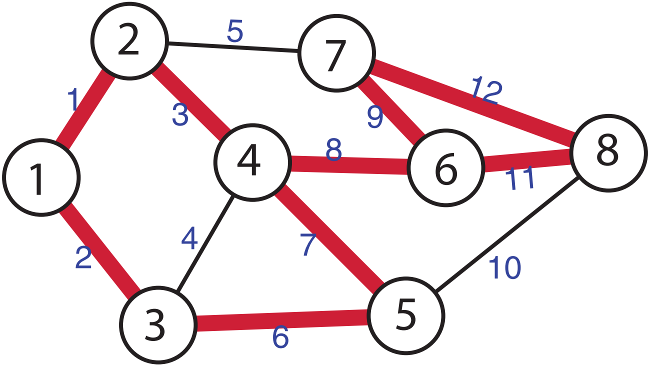

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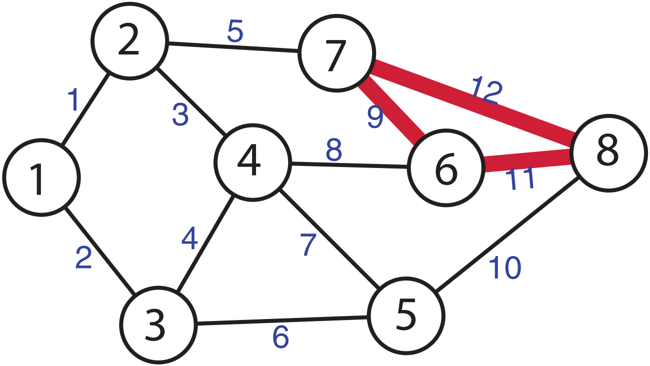

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$$\mathcal{I} = \{X \subseteq V : |X \cap V_i| \leq k_i \text{ for all } i = 1, \ldots, \ell\}. \quad (6.10)$$

where $k_1, \ldots, k_\ell$ are fixed "limit" parameters, $k_i \geq 0$. Then $M = (V, \mathcal{I})$ is a matroid.

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- Note that a k-uniform matroid is a trivial example of a partition matroid with $\ell = 1$, $V_1 = V$, and $k_1 = k$.

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- Note that a k-uniform matroid is a trivial example of a partition matroid with $\ell = 1$, $V_1 = V$, and $k_1 = k$.
- Parameters associated with a partition matroid: ℓ and $k_1, k_2, \ldots, k_\ell$ although often the k_i's are all the same.

Partition Matroid

- Let V be our ground set.
- Let $V = V_1 \cup V_2 \cup \cdots \cup V_\ell$ be a partition of V into ℓ blocks (i.e., disjoint sets). Define a set of subsets of V as

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- If $X, Y \in \mathcal{I}$ with $|Y| > |X|$, then there must be at least one i with $|Y \cap V_i| > |X \cap V_i|$. Therefore, adding one element $e \in V_i \cap (Y \setminus X)$ to X won't break independence.

Partition Matroid

Ground set of objects, $V = \Bigg\{$

$\Bigg\}$

Partition Matroid

Partition of V into six blocks, $V_1, V_2, \dots, V_6$

Partition Matroid

Limit associated with each block, $\{k_1, k_2, \ldots, k_6\}$

Partition Matroid

Independent subset but not maximally independent.

Partition Matroid

Maximally independent subset, what is called a base.

Partition Matroid

Not independent since over limit in set six.

Matroids - rank function is submodular

Lemma 6.5.1

The rank function $r : 2^E \to \mathbb{Z}_+$ *of a matroid is submodular, that is* $r(A) + r(B) \geq r(A \cup B) + r(A \cap B)$

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Proof.

1. Let $X \in \mathcal{I}$ be an inclusionwise maximal set with $X \subseteq A \cap B$

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2. Let $Y \in \mathcal{I}$ be inclusionwise maximal set with $X \subseteq Y \subseteq A \cup B$. *We can find such a* $Y \supseteq X$ *because the following. Let* $Y' \in \mathcal{I}$ *be any inclusionwise maximal set with* $Y' \subseteq A \cup B$*, which might not have* $X \subseteq Y'$*. Starting from* $Y \leftarrow X \subseteq A \cup B$*, since* $|Y'| \geq |X|$*, there exists a* $y \in Y' \setminus X \subseteq A \cup B$ *such that* $X + y \in \mathcal{I}$ *but since* $y \in A \cup B$*, also* $X + y \in A \cup B$ *— we then add* y *to* Y*. We can keep doing this while* $|Y'| > |X|$ *since this is a matroid. We end up with an inclusionwise maximal set* Y *with* $Y \in \mathcal{I}$ *and* $X \subseteq Y$*.*

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3. Since M is a matroid, we know that $r(A \cap B) = r(X) = |X|$, and $r(A \cup B) = r(Y) = |Y|$. Also, for any $U \in \mathcal{I}$, $r(A) \geq |A \cap U|$.

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$$\geq |X| + |Y| = r(A \cap B) + r(A \cup B) \tag{6.13}$$

□

A matroid is defined from its rank function

Theorem 6.5.2 (Matroid from rank)

Let E be a set and let $r : 2^E \to \mathbb{Z}_+$ be a function. Then $r(\cdot)$ defines a matroid with r being its rank function if and only if for all $A, B \subseteq E$:

(R1) $\forall A \subseteq E \;\; 0 \leq r(A) \leq |A|$ *(non-negative cardinality bounded)*

(R2) $r(A) \leq r(B)$ *whenever* $A \subseteq B \subseteq E$ *(monotone non-decreasing)*

(R3) $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$ *for all* $A, B \subseteq E$ *(submodular)*

- From above, $r(\emptyset) = 0$. Let $v \notin A$, then by monotonicity and submodularity, $r(A) \leq r(A \cup \{v\}) \leq r(A) + r(\{v\})$ which gives only two possible values to $r(A \cup \{v\})$, namely $r(A)$ or $r(A) + 1$.

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- Thus, submodularity, non-negative monotone non-decreasing, and unit increment of rank is necessary and sufficient to define a matroid.

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- Thus, submodularity, non-negative monotone non-decreasing, and unit increment of rank is necessary and sufficient to define a matroid.
- Can refer to matroid as (E, r), E is ground set, r is rank function.

Matroids from rank

Proof of Theorem 6.5.2 (matroid from rank).

- Given a matroid $M = (E, \mathcal{I})$, we see its rank function as defined in Eq. 6.7 satisfies (R1), (R2), and, as we saw in Lemma 6.5.1, (R3) too.

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$$\geq |Y| - |Y \setminus X| \tag{6.15}$$

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$$= |X| \tag{6.16}$$

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- First, $\emptyset \in \mathcal{I}$.
- Also, if $Y \in \mathcal{I}$ and $X \subseteq Y$ then by submodularity,

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$$= |X| \tag{6.16}$$

implying $r(X) = |X|$, and thus $X \in \mathcal{I}$.

...

Matroids from rank

Proof of Theorem 6.5.2 (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $1 \leq k \leq |B|$).

□

Matroids from rank

Proof of Theorem 6.5.2 (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $1 \leq k \leq |B|$).
- Suppose, to the contrary, that $\forall b \in B \setminus A$, $A + b \notin \mathcal{I}$, which means for all such b, $r(A + b) = r(A) = |A| < |A| + 1$. Then

□

Matroids from rank

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$$r(B) \leq r(A \cup B) \tag{6.17}$$

□

Matroids from rank

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$$r(B) \leq r(A \cup B) \tag{6.17}$$

$$\leq r(A \cup (B \setminus \{b_1\})) + r(A \cup \{b_1\}) - r(A) \tag{6.18}$$

□

Matroids from rank

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- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $1 \leq k \leq |B|$).
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$$= r(A \cup (B \setminus \{b_1\}) \tag{6.19}$$

□

Matroids from rank

Proof of Theorem 6.5.2 (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $1 \leq k \leq |B|$).
- Suppose, to the contrary, that $\forall b \in B \setminus A$, $A + b \notin \mathcal{I}$, which means for all such b, $r(A + b) = r(A) = |A| < |A| + 1$. Then

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$$= r(A \cup (B \setminus \{b_1\}) \tag{6.19}$$

$$\leq r(A \cup (B \setminus \{b_1, b_2\})) + r(A \cup \{b_2\}) - r(A) \tag{6.20}$$

□

Matroids from rank

Proof of Theorem 6.5.2 (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \dots, b_k\}$ (note $1 \leq k \leq |B|$).
- Suppose, to the contrary, that $\forall b \in B \setminus A$, $A + b \notin \mathcal{I}$, which means for all such b, $r(A + b) = r(A) = |A| < |A| + 1$. Then

$$\begin{aligned} r(B) &\leq r(A \cup B) && (6.17)\\ &\leq r(A \cup (B \setminus \{b_1\})) + r(A \cup \{b_1\}) - r(A) && (6.18)\\ &= r(A \cup (B \setminus \{b_1\}) && (6.19)\\ &\leq r(A \cup (B \setminus \{b_1, b_2\})) + r(A \cup \{b_2\}) - r(A) && (6.20)\\ &= r(A \cup (B \setminus \{b_1, b_2\})) && (6.21) \end{aligned}$$

□

Matroids from rank

Proof of Theorem 6.5.2 (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \dots, b_k\}$ (note $1 \le k \le |B|$).
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$$
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&\le r(A \cup (B \setminus \{b_1\})) + r(A \cup \{b_1\}) - r(A) && (6.18)\\
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&\le r(A \cup (B \setminus \{b_1, b_2\})) + r(A \cup \{b_2\}) - r(A) && (6.20)\\
&= r(A \cup (B \setminus \{b_1, b_2\})) && (6.21)\\
&\le \dots \le r(A) = |A| < |B| && (6.22)
\end{aligned}
$$

□

Matroids from rank

Proof of Theorem 6.5.2 (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \dots, b_k\}$ (note $1 \leq k \leq |B|$).
- Suppose, to the contrary, that $\forall b \in B \setminus A$, $A + b \notin \mathcal{I}$, which means for all such b, $r(A + b) = r(A) = |A| < |A| + 1$. Then

$$
\begin{aligned}
r(B) &\leq r(A \cup B) && (6.17)\\
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$$

giving a contradiction since $B \in \mathcal{I}$. □

Matroids from rank II

Another way of using function r to define a matroid.

Theorem 6.5.3 (Matroid from rank II)

Let E be a finite set and let $r : 2^E \to \mathbb{Z}_+$ be a function. Then $r(\cdot)$ defines a matroid with r being its rank function if and only if for all $X \subseteq E$, and $x, y \in E$:

(R1') $r(\emptyset) = 0$;

(R2') $r(X) \leq r(X \cup \{y\}) \leq r(X) + 1$;

(R3') *If* $r(X \cup \{x\}) = r(X \cup \{y\}) = r(X)$, *then* $r(X \cup \{x, y\}) = r(X)$.

Matroids by submodular functions

Theorem 6.5.4 (Matroid by submodular functions)

Let $f : 2^E \to \mathbb{Z}$ be a integer valued monotone non-decreasing submodular function. Define a set of sets as follows:

$$\mathcal{C}(f) = \Big\{ C \subseteq E : C \text{ is non-empty, is inclusionwise-minimal, and has } f(C) < |C| \Big\} \tag{6.23}$$

Then $\mathcal{C}(f)$ is the collection of circuits of a matroid on E.

Inclusionwise-minimal in this case means that if $C \in \mathcal{C}(f)$, then there exists no $C' \subset C$ with $C' \in \mathcal{C}(f)$ (i.e., $C' \subset C$ would either be empty or have $f(C') \geq |C'|$). Also, recall inclusionwise-minimal in Definition 6.3.10, the definition of a circuit.

Summarizing: Many ways to define a Matroid

Summarizing what we've so far seen, we saw that it is possible to uniquely define a matroid based on any of:

- Independence (define the independent sets).

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- Rank axioms (normalized, monotone, cardinality bounded, non-negative integral, submodular)
- Matroids by integral submodular functions.

Maximization problems for matroids

- Given a matroid $M = (E, \mathcal{I})$ and a modular value function $c : E \to \mathbb{R}$, the task is to find an $X \in \mathcal{I}$ such that $c(X) = \sum_{x \in X} c(x)$ is maximum.
- This seems remarkably similar to the max spanning tree problem.

Minimization problems for matroids

- Given a matroid $M = (E, \mathcal{I})$ and a modular cost function $c : E \to \mathbb{R}$, the task is to find a basis $B \in \mathcal{B}$ such that $c(B)$ is minimized.
- This sounds like a set cover problem (find the minimum cost covering set of sets).

Partition Matroid

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- A partition matroids rank function:

$$r(A) = \sum_{i=1}^{\ell} \min(|A \cap V_i|, k_i) \tag{6.24}$$

which we also immediately see is submodular using properties we spoke about last week. That is:

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  2. $\min(\text{submodular}(A), k_i)$ is submodular in A since $|A \cap V_i|$ is monotone.
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- $r(A)$ is also non-negative integral monotone non-decreasing, so it defines a matroid (the partition matroid).

From 2-partition matroid rank to truncated matroid rank

- Example: 2-partition matroid rank function: Given natural numbers $a, b \in \mathbb{Z}_+$ with $a < b$, and any set $R \subseteq V$ with $|R| = b$.

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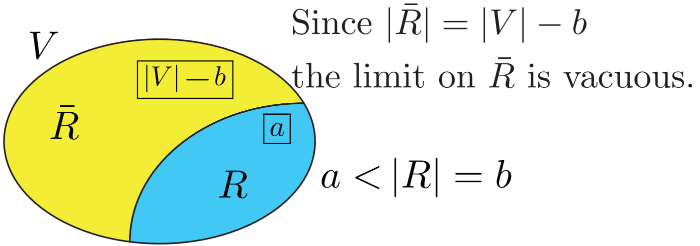
- Figure showing partition blocks and partition matroid limits.



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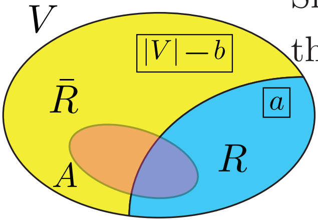

Since $|\bar{R}| = |V| - b$ the limit on $\bar{R}$ is vacuous.

$a < |R| = b$

Truncated Matroid Rank Function

- Define truncated matroid rank function. Start with 2-partition matroid rank $r(A) = \min(|A \cap R|, a) + \min(|A \cap \bar{R}|, |\bar{R}|)$, $a < b$. Define:

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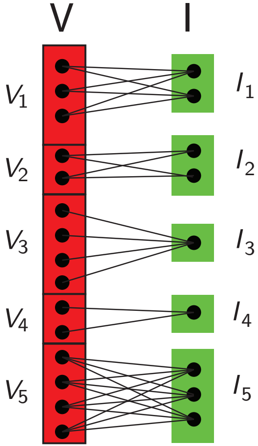
- For any B with $|B \cap R| = \ell$, with $a \leq \ell \leq b$, $f_R(B) = a + b - \ell$.
- R, the set with minimum valuation amongst size-b sets, is hidden within an exponentially larger set of size-b sets with larger valuation.

Partition Matroid, rank as matching

- A partition matroid can be viewed using a bipartite graph.
- Letting V denote the ground set, and $V_1, V_2, \dots$ the partition, the bipartite graph is $G = (V, I, E)$ where V is the ground set, I is a set of "indices", and E is the set of edges.
- $I = (I_1, I_2, \dots, I_\ell)$ is a set of $k = \sum_{i=1}^{\ell} k_i$ nodes, grouped into ℓ clusters, where there are k_i nodes in the i^{th} group I_i, and $|I_i| = k_i$.
- $(v, i) \in E(G)$ iff $v \in V_j$ and $i \in I_j$.

Partition Matroid, rank as matching

- Example where $\ell = 5$, $(k_1, k_2, k_3, k_4, k_5) = (2, 2, 1, 1, 3)$.



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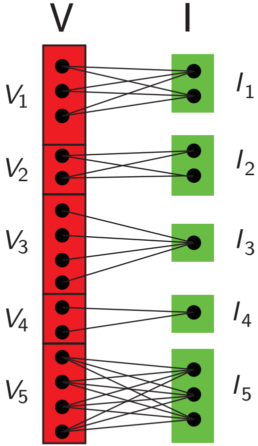

- Recall, $\Gamma : 2^V \to \mathbb{R}$ as the neighbor function in a bipartite graph, the neighbors of X is defined as $\Gamma(X) = \{v \in V(G) \setminus X : E(X, \{v\}) \neq \emptyset\}$, and recall that $|\Gamma(X)|$ is submodular.

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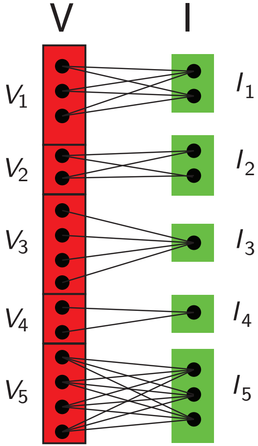

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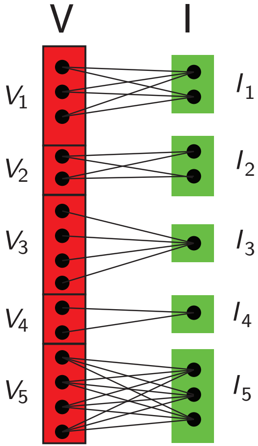

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- Here, for $X \subseteq V$, we have $\Gamma(X) = \{i \in I : (v, i) \in E(G) \text{ and } v \in X\}$.
- For such a constructed bipartite graph, the rank function of a partition matroid is $r(X) = \sum_{i=1}^{\ell} \min(|X \cap V_i|, k_i) =$ the maximum matching involving X.