

# Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture 6 —

[http://www.ee.washington.edu/people/faculty/bilmes/classes/ee563\\_spring\\_2018/](http://www.ee.washington.edu/people/faculty/bilmes/classes/ee563_spring_2018/)

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$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

$= f(A) + 2f(C) + f(B) = f(A) + f(C) + f(B) = f(A \cap B)$



## Cumulative Outstanding Reading

- Read chapter 1 from Fujishige's book.
- Read chapter 2 from Fujishige's book.

## Announcements, Assignments, and Reminders

- If you have any questions about anything, please ask then via our discussion board ([https://canvas.uw.edu/courses/1216339/discussion\\_topics](https://canvas.uw.edu/courses/1216339/discussion_topics)).

## Class Road Map - EE563

- L1(3/26): Motivation, Applications, & Basic Definitions,
- L2(3/28): Machine Learning Apps (diversity, complexity, parameter, learning target, surrogate).
- L3(4/2): Info theory exs, more apps, definitions, graph/combinatorial examples
- L4(4/4): Graph and Combinatorial Examples, Matrix Rank, Examples and Properties, visualizations
- L5(4/9): More Examples/Properties/ Other Submodular Defs., Independence,
- L6(4/11): Matroids, Matroid Examples, Matroid Rank, Partition/Laminar Matroids
- L7(4/16): Laminar Matroids, System of Distinct Reprs, Transversals, Transversal Matroid, Matroid Representation, Dual Matroids
- L8(4/18):
- L9(4/23):
- L10(4/25):
- L11(4/30):
- L12(5/2):
- L13(5/7):
- L14(5/9):
- L15(5/14):
- L16(5/16):
- L17(5/21):
- L18(5/23):
- L-(5/28): Memorial Day (holiday)
- L19(5/30):
- L21(6/4): Final Presentations maximization.

Last day of instruction, June 1st. Finals Week: June 2-8, 2018.

## Composition of non-decreasing submodular and non-decreasing concave

### Theorem 6.2.1

Given two functions, one defined on sets

$$f : 2^V \rightarrow \mathbb{R} \quad (6.1)$$

and another continuous valued one:

$$\phi : \mathbb{R} \rightarrow \mathbb{R} \quad (6.2)$$

the composition formed as  $h = \phi \circ f : 2^V \rightarrow \mathbb{R}$  (defined as  $h(S) = \phi(f(S))$ ) is nondecreasing submodular, if  $\phi$  is non-decreasing concave and  $f$  is nondecreasing submodular.

## Monotone difference of two functions

Let  $f$  and  $g$  both be submodular functions on subsets of  $V$  and let  $(f - g)(\cdot)$  be either monotone non-decreasing or monotone non-increasing  
Then  $h : 2^V \rightarrow \mathbb{R}$  defined by

$$h(A) = \min(f(A), g(A)) \quad (6.1)$$

is submodular.

### Proof.

If  $h(A)$  agrees with  $f$  on **both**  $X$  and  $Y$  (or  $g$  on both  $X$  and  $Y$ ), and since

$$h(X) + h(Y) = f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y) \quad (6.2)$$

or

$$h(X) + h(Y) = g(X) + g(Y) \geq g(X \cup Y) + g(X \cap Y), \quad (6.3)$$

the result (Equation 6.1 being submodular) follows since

$$\begin{aligned} f(X) + f(Y) &\geq \min(f(X \cup Y), g(X \cup Y)) + \min(f(X \cap Y), g(X \cap Y)) \\ g(X) + g(Y) &\geq \min(f(X \cup Y), g(X \cup Y)) + \min(f(X \cap Y), g(X \cap Y)) \end{aligned} \quad (6.4)$$

...

## Arbitrary functions: difference between submodular funcs.

### Theorem 6.2.1

Given an arbitrary set function  $h$ , it can be expressed as a difference between two submodular functions (i.e.,  $\forall h \in 2^V \rightarrow \mathbb{R}$ ,  $\exists f, g$  s.t.  $\forall A, h(A) = f(A) - g(A)$  where both  $f$  and  $g$  are submodular).

### Proof.

Let  $h$  be given and arbitrary, and define:

$$\alpha \triangleq \min_{X, Y: X \not\subseteq Y, Y \not\subseteq X} (h(X) + h(Y) - h(X \cup Y) - h(X \cap Y)) \quad (6.4)$$

If  $\alpha \geq 0$  then  $h$  is submodular, so by assumption  $\alpha < 0$ . Now let  $f$  be an arbitrary **strict** submodular function and define

$$\beta \triangleq \min_{X, Y: X \not\subseteq Y, Y \not\subseteq X} (f(X) + f(Y) - f(X \cup Y) - f(X \cap Y)). \quad (6.5)$$

Strict means that  $\beta > 0$ . ...

## Many (Equivalent) Definitions of Submodularity

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B), \quad \forall A, B \subseteq V \quad (6.16)$$

$$f(j|S) \geq f(j|T), \quad \forall S \subseteq T \subseteq V, \text{ with } j \in V \setminus T \quad (6.17)$$

$$f(C|S) \geq f(C|T), \quad \forall S \subseteq T \subseteq V, \text{ with } C \subseteq V \setminus T \quad (6.18)$$

$$f(j|S) \geq f(j|S \cup \{k\}), \quad \forall S \subseteq V \text{ with } j \in V \setminus (S \cup \{k\}) \quad (6.19)$$

$$f(A \cup B|A \cap B) \leq f(A|A \cap B) + f(B|A \cap B), \quad \forall A, B \subseteq V \quad (6.20)$$

$$f(T) \leq f(S) + \sum_{j \in T \setminus S} f(j|S) - \sum_{j \in S \setminus T} f(j|S \cup T - \{j\}), \quad \forall S, T \subseteq V \quad (6.21)$$

$$f(T) \leq f(S) + \sum_{j \in T \setminus S} f(j|S), \quad \forall S \subseteq T \subseteq V \quad (6.22)$$

$$f(T) \leq f(S) - \sum_{j \in S \setminus T} f(j|S \setminus \{j\}) + \sum_{j \in T \setminus S} f(j|S \cap T) \quad \forall S, T \subseteq V \quad (6.23)$$

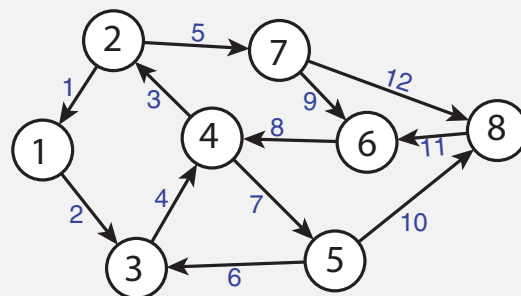
$$f(T) \leq f(S) - \sum_{j \in S \setminus T} f(j|S \setminus \{j\}), \quad \forall T \subseteq S \subseteq V \quad (6.24)$$

## On Rank

- Let  $\text{rank} : 2^V \rightarrow \mathbb{Z}_+$  be the rank function.
- In general,  $\text{rank}(A) \leq |A|$ , and vectors in  $A$  are linearly independent if and only if  $\text{rank}(A) = |A|$ .
- If  $A, B$  are such that  $\text{rank}(A) = |A|$  and  $\text{rank}(B) = |B|$ , with  $|A| < |B|$ , then the space spanned by  $B$  is greater, and we can find a vector in  $B$  that is linearly independent of the space spanned by vectors in  $A$ .
- To stress this point, note that the above condition is  $|A| < |B|$ , **not**  $A \subseteq B$  which is sufficient (to be able to find an independent vector) but not required.
- In other words, given  $A, B$  with  $\text{rank}(A) = |A|$  &  $\text{rank}(B) = |B|$ , then  $|A| < |B| \Leftrightarrow \exists$  an  $b \in B$  such that  $\text{rank}(A \cup \{b\}) = |A| + 1$ .

## Spanning trees/forests & incidence matrices

- A directed version of the graph (right) and its adjacency matrix (below).
- Orientation can be arbitrary.
- Note, rank of this matrix is 7.



$$\begin{array}{c}
 \begin{array}{cccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
 \begin{array}{l}
 1 \\
 2 \\
 3 \\
 4 \\
 5 \\
 6 \\
 7 \\
 8
 \end{array}
 \left( \begin{array}{cccccccccccc}
 -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & 0 \\
 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1
 \end{array} \right)
 \end{array}
 \end{array}$$

## From Matrix Rank $\rightarrow$ Matroid

- So  $V$  is set of column vector indices of a matrix.
- Let  $\mathcal{I} = \{I_1, I_2, \dots\}$  be a set of all subsets of  $V$  such that for any  $I \in \mathcal{I}$ , the vectors indexed by  $I$  are linearly independent.
- Given a set  $B \in \mathcal{I}$  of linearly independent vectors, then any subset  $A \subseteq B$  is also linearly independent. Hence,  $\mathcal{I}$  is down-closed or “subclusive”, under subsets. In other words,

$$A \subseteq B \text{ and } B \in \mathcal{I} \Rightarrow A \in \mathcal{I} \quad (6.1)$$

- **maxInd**: Inclusionwise maximal independent subsets (i.e., the set of **bases** of) of any set  $B \subseteq V$  defined as:

$$\text{maxInd}(B) \triangleq \{A \subseteq B : A \in \mathcal{I} \text{ and } \forall v \in B \setminus A, A \cup \{v\} \notin \mathcal{I}\} \quad (6.2)$$

- Given any set  $B \subseteq V$  of vectors, all maximal (by set inclusion) subsets of linearly independent vectors are the same size. That is, for all  $B \subseteq V$ ,

$$\forall A_1, A_2 \in \text{maxInd}(B), \quad |A_1| = |A_2| = \text{rank}(B) \quad (6.3)$$

## From Matrix Rank $\rightarrow$ Matroid

- Let  $\mathcal{I} = \{I_1, I_2, \dots\}$  be the set of sets as described above.
- Thus, for all  $I \in \mathcal{I}$ , the matrix rank function has the property

$$r(I) = |I| \quad (6.4)$$

and for any  $B \notin \mathcal{I}$ ,

$$r(B) = \max \{|A| : A \subseteq B \text{ and } A \in \mathcal{I}\} < |B| \quad (6.5)$$

Since all maximally independent subsets of a set are the same size, the rank function is well defined.

# Matroid

- Matroids abstract the notion of linear independence of a set of vectors to general algebraic properties.
- In a matroid, there is an underlying **ground set**, say  $E$  (or  $V$ ), and a collection of subsets  $\mathcal{I} = \{I_1, I_2, \dots\}$  of  $E$  that correspond to independent elements.
- There are many definitions of matroids that are mathematically equivalent, we'll see some of them here.

# Independence System

## Definition 6.3.1 (set system)

A (finite) ground set  $E$  and a set of subsets of  $E$ ,  $\emptyset \neq \mathcal{I} \subseteq 2^E$  is called a set system, notated  $(E, \mathcal{I})$ .

- Set systems can be arbitrarily complex since, as stated, there is no systematic method (besides exponential-cost exhaustive search) to determine if a given set  $S \subseteq E$  has  $S \in \mathcal{I}$ .
- One useful property is “heredity.” Namely, a set system is a hereditary set system if for any  $A \subset B \in \mathcal{I}$ , we have that  $A \in \mathcal{I}$ .

# Independence System

## Definition 6.3.2 (independence (or hereditary) system)

A set system  $(V, \mathcal{I})$  is an independence system if

$$\emptyset \in \mathcal{I} \quad (\text{emptyset containing}) \quad (I1)$$

and

$$\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I} \quad (\text{subclusive}) \quad (I2)$$

- Property (I2) called “down monotone,” “down closed,” or “subclusive”
- Example:  $E = \{1, 2, 3, 4\}$ . With  $\mathcal{I} = \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 4\}\}$ .
- Then  $(E, \mathcal{I})$  is a set system, but not an independence system since it is not down closed (e.g., we have  $\{1, 2\} \in \mathcal{I}$  but not  $\{2\} \in \mathcal{I}$ ).
- With  $\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ , then  $(E, \mathcal{I})$  is now an independence (hereditary) system.

# Independence System

$$\begin{array}{r}
 1 \\
 2 \\
 3
 \end{array}
 \begin{pmatrix}
 0 & 0 & 1 & 1 & 2 & 1 & 3 & 1 \\
 0 & 1 & 1 & 0 & 2 & 0 & 2 & 4 \\
 1 & 1 & 1 & 0 & 0 & 3 & 1 & 5
 \end{pmatrix}
 =
 \begin{pmatrix}
 | & | & | & | & | & | & | & | \\
 x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\
 | & | & | & | & | & | & | & |
 \end{pmatrix}
 \quad (6.6)$$

- Given any set of linearly independent vectors  $A$ , any subset  $B \subset A$  will also be linearly independent.
- Given any forest  $G_f$  that is an edge-induced sub-graph of a graph  $G$ , any sub-graph of  $G_f$  is also a forest.
- So these both constitute independence systems.



# Matroid

Independent set definition of a matroid is perhaps most natural. Note, if  $J \in \mathcal{I}$ , then  $J$  is said to be an **independent set**.

## Definition 6.3.3 (Matroid)

A set system  $(E, \mathcal{I})$  is a **Matroid** if

- (I1)  $\emptyset \in \mathcal{I}$
- (I2)  $\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}$  (down-closed or subclusive)
- (I3)  $\forall I, J \in \mathcal{I}$ , with  $|I| = |J| + 1$ , then there exists  $x \in I \setminus J$  such that  $J \cup \{x\} \in \mathcal{I}$ .

Why is (I1) is not redundant given (I2)? Because without (I1) could have a non-matroid where  $\mathcal{I} = \{\}$ .

# On Matroids

- Abstract properties of linear dependence (Hassler Whitney, 1935), but already then found instances of objects with those properties not based on a matrix.
- Takeo Nakasawa, 1935, also early work.
- Forgotten for 20 years until mid 1950s.
- Matroids are powerful and flexible combinatorial objects.
- The rank function of a matroid is already a very powerful submodular function (perhaps all we need for many problems).
- Understanding matroids crucial for understanding submodularity.
- Matroid independent sets (i.e.,  $A$  s.t.  $r(A) = |A|$ ) are useful constraint set, and fast algorithms for submodular optimization subject to one (or more) matroid independence constraints exist.
- Crapo & Rota preferred the term “combinatorial geometry”, or more specifically a “pregeometry” and said that pregeometries are “often described by the ineffably cacaphonic [sic] term ‘matroid’, which we prefer to avoid in favor of the term ‘pregeometry’.”

# Matroid

Slight modification (non unit increment) that is equivalent.

## Definition 6.3.4 (Matroid-II)

A set system  $(E, \mathcal{I})$  is a **Matroid** if

- (I1')  $\emptyset \in \mathcal{I}$
- (I2')  $\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}$  (down-closed or subclusive)
- (I3')  $\forall I, J \in \mathcal{I}$ , with  $|I| > |J|$ , then there exists  $x \in I \setminus J$  such that  $J \cup \{x\} \in \mathcal{I}$

Note (I1)=(I1'), (I2)=(I2'), and we get (I3) $\equiv$ (I3') using induction.

# Matroids, independent sets, and bases

- **Independent sets:** Given a matroid  $M = (E, \mathcal{I})$ , a subset  $A \subseteq E$  is called **independent** if  $A \in \mathcal{I}$  and otherwise  $A$  is called **dependent**.
- **A base of  $U \subseteq E$ :** For  $U \subseteq E$ , a subset  $B \subseteq U$  is called a **base** of  $U$  if  $B$  is inclusionwise maximally independent subset of  $U$ . That is,  $B \in \mathcal{I}$  and there is no  $Z \in \mathcal{I}$  with  $B \subset Z \subseteq U$ .
- **A base of a matroid:** If  $U = E$ , then a "base of  $E$ " is just called a **base** of the matroid  $M$  (this corresponds to a **basis** in a linear space, or a **spanning forest** in a graph, or a **spanning tree** in a connected graph).

## Matroids - important property

### Proposition 6.3.5

In a matroid  $M = (E, \mathcal{I})$ , for any  $U \subseteq E(M)$ , any two bases of  $U$  have the same size.

- In matrix terms, given a set of vectors  $U$ , all sets of independent vectors that span the space spanned by  $U$  have the same size.
- In fact, under (I1),(I2), this condition is equivalent to (I3). **Exercise:** show the following is equivalent to the above.

### Definition 6.3.6 (Matroid)

A set system  $(V, \mathcal{I})$  is a **Matroid** if

- (I1')  $\emptyset \in \mathcal{I}$  (emptyset containing)
- (I2')  $\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}$  (down-closed or subclusive)
- (I3')  $\forall X \subseteq V$ , and  $I_1, I_2 \in \max \text{Ind}(X)$ , we have  $|I_1| = |I_2|$  (all maximally independent subsets of  $X$  have the same size).

## Matroids - rank

- Thus, in any matroid  $M = (E, \mathcal{I})$ ,  $\forall U \subseteq E(M)$ , any two bases of  $U$  have the same size.
- The common size of all the **bases** of  $U$  is called the rank of  $U$ , denoted  $r_M(U)$  or just  $r(U)$  when the matroid in equation is unambiguous.
- $r(E) = r_{(E, \mathcal{I})}$  is the rank of the matroid, and is the common size of all the bases of the matroid.
- We can a bit more formally define the rank function this way.

### Definition 6.3.7 (matroid rank function)

The rank function of a matroid is a function  $r : 2^E \rightarrow \mathbb{Z}_+$  defined by

$$r(A) = \max \{ |X| : X \subseteq A, X \in \mathcal{I} \} = \max_{X \in \mathcal{I}} |A \cap X| \quad (6.7)$$

- From the above, we immediately see that  $r(A) \leq |A|$ .
- Moreover, if  $r(A) = |A|$ , then  $A \in \mathcal{I}$ , meaning  $A$  is independent (in this case,  $A$  is a **self base**).

# Matroids, other definitions using matroid rank $r : 2^V \rightarrow \mathbb{Z}_+$

## Definition 6.3.8 (closed/flat/subspace)

A subset  $A \subseteq E$  is **closed** (equivalently, a **flat** or a **subspace**) of matroid  $M$  if for all  $x \in E \setminus A$ ,  $r(A \cup \{x\}) = r(A) + 1$ .

Definition: A **hyperplane** is a flat of rank  $r(M) - 1$ .

## Definition 6.3.9 (closure)

Given  $A \subseteq E$ , the **closure** (or **span**) of  $A$ , is defined by  $\text{span}(A) = \{b \in E : r(A \cup \{b\}) = r(A)\}$ .

Therefore, a closed set  $A$  has  $\text{span}(A) = A$ .

## Definition 6.3.10 (circuit)

A subset  $A \subseteq E$  is **circuit** or a **cycle** if it is an inclusionwise-minimal dependent set (i.e., if  $r(A) < |A|$  and for any  $a \in A$ ,  $r(A \setminus \{a\}) = |A| - 1$ ).

# Matroids by bases

In general, besides independent sets and rank functions, there are other equivalent ways to characterize matroids.

## Theorem 6.3.11 (Matroid (by bases))

Let  $E$  be a set and  $\mathcal{B}$  be a nonempty collection of subsets of  $E$ . Then the following are equivalent.

- ①  $\mathcal{B}$  is the collection of bases of a matroid;
- ② if  $B, B' \in \mathcal{B}$ , and  $x \in B' \setminus B$ , then  $B' - x + y \in \mathcal{B}$  for some  $y \in B \setminus B'$ .
- ③ If  $B, B' \in \mathcal{B}$ , and  $x \in B' \setminus B$ , then  $B - y + x \in \mathcal{B}$  for some  $y \in B \setminus B'$ .

Properties 2 and 3 are called “exchange properties.”

Proof here is omitted but think about this for a moment in terms of linear spaces and matrices, and (alternatively) spanning trees.

## Matroids by circuits

A set is independent if and only if it contains no circuit. Therefore, it is not surprising that circuits can also characterize a matroid.

### Theorem 6.3.12 (Matroid by circuits)

Let  $E$  be a set and  $\mathcal{C}$  be a collection of subsets of  $E$  that satisfy the following three properties:

- 1 (C1):  $\emptyset \notin \mathcal{C}$
- 2 (C2): if  $C_1, C_2 \in \mathcal{C}$  and  $C_1 \subseteq C_2$ , then  $C_1 = C_2$ .
- 3 (C3): if  $C_1, C_2 \in \mathcal{C}$  with  $C_1 \neq C_2$ , and  $e \in C_1 \cap C_2$ , then there exists a  $C_3 \in \mathcal{C}$  such that  $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$ .

## Matroids by circuits

Several circuit definitions for matroids.

### Theorem 6.3.13 (Matroid by circuits)

Let  $E$  be a set and  $\mathcal{C}$  be a collection of nonempty subsets of  $E$ , such that no two sets in  $\mathcal{C}$  are contained in each other. Then the following are equivalent.

- 1  $\mathcal{C}$  is the collection of circuits of a matroid;
- 2 if  $C, C' \in \mathcal{C}$ , and  $x \in C \cap C'$ , then  $(C \cup C') \setminus \{x\}$  contains a set in  $\mathcal{C}$ ;
- 3 if  $C, C' \in \mathcal{C}$ , and  $x \in C \cap C'$ , and  $y \in C \setminus C'$ , then  $(C \cup C') \setminus \{x\}$  contains a set in  $\mathcal{C}$  containing  $y$ ;

Again, think about this for a moment in terms of linear spaces and matrices, and spanning trees.

## Uniform Matroid

- Given  $E$ , consider  $\mathcal{I}$  to be all subsets of  $E$  that are at most size  $k$ . That is  $\mathcal{I} = \{A \subseteq E : |A| \leq k\}$ .
- Then  $(E, \mathcal{I})$  is a matroid called a  $k$ -uniform matroid.
- Note, if  $I, J \in \mathcal{I}$ , and  $|I| < |J| \leq k$ , and  $j \in J$  such that  $j \notin I$ , then  $j$  is such that  $|I + j| \leq k$  and so  $I + j \in \mathcal{I}$ .
- Rank function

$$r(A) = \begin{cases} |A| & \text{if } |A| \leq k \\ k & \text{if } |A| > k \end{cases} \quad (6.8)$$

- Note, this function is submodular. Not surprising since  $r(A) = \min(|A|, k)$  which is a non-decreasing concave function applied to a modular function.
- Closure function

$$\text{span}(A) = \begin{cases} A & \text{if } |A| < k, \\ E & \text{if } |A| \geq k, \end{cases} \quad (6.9)$$

- A “free” matroid sets  $k = |E|$ , so everything is independent.

## Linear (or Matric) Matroid

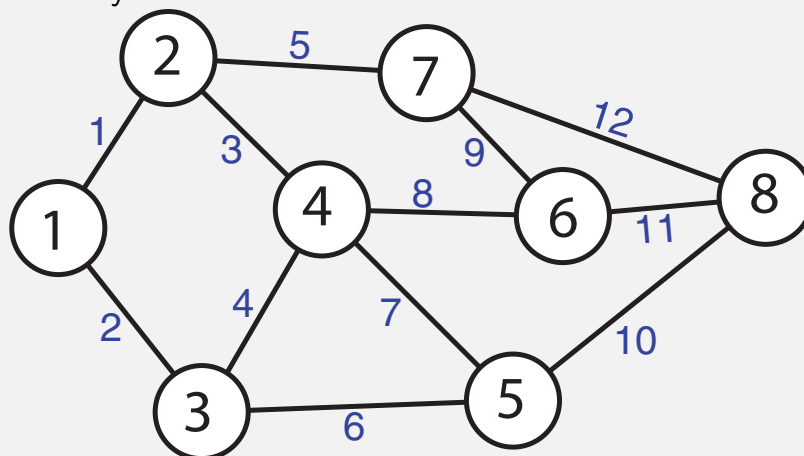
- Let  $\mathbf{X}$  be an  $n \times m$  matrix and  $E = \{1, \dots, m\}$
- Let  $\mathcal{I}$  consists of subsets of  $E$  such that if  $A \in \mathcal{I}$ , and  $A = \{a_1, a_2, \dots, a_k\}$  then the vectors  $x_{a_1}, x_{a_2}, \dots, x_{a_k}$  are linearly independent.
- the rank function is just the rank of the space spanned by the corresponding set of vectors.
- rank is submodular, it is intuitive that it satisfies the diminishing returns property (a given vector can only become linearly dependent in a greater context, thereby no longer contributing to rank).
- Called both linear matroids and matric matroids.

## Cycle Matroid of a graph: Graphic Matroids

- Let  $G = (V, E)$  be a graph. Consider  $(E, \mathcal{I})$  where the edges of the graph  $E$  are the ground set and  $A \in \mathcal{I}$  if the edge-induced graph  $G(V, A)$  by  $A$  does not contain any cycle.
- Then  $M = (E, \mathcal{I})$  is a matroid.
- $\mathcal{I}$  contains all forests.
- Bases are spanning forests (spanning trees if  $G$  is connected).
- Rank function  $r(A)$  is the size of the largest spanning forest contained in  $G(V, A)$ .
- Recall from earlier,  $r(A) = |V(G)| - k_G(A)$ , where for  $A \subseteq E(G)$ , we define  $k_G(A)$  as the number of connected components of the edge-induced spanning subgraph  $(V(G), A)$ , and that  $k_G(A)$  is supermodular, so  $|V(G)| - k_G(A)$  is submodular.
- Closure function adds all edges between the vertices adjacent to any edge in  $A$ . Closure of a spanning forest is  $G$ .

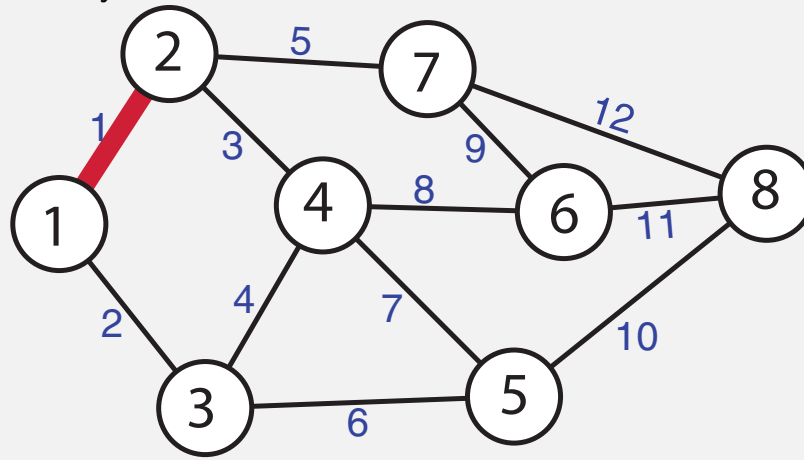
## Example: graphic matroid

- A graph defines a matroid on edge sets, independent sets are those without a cycle.



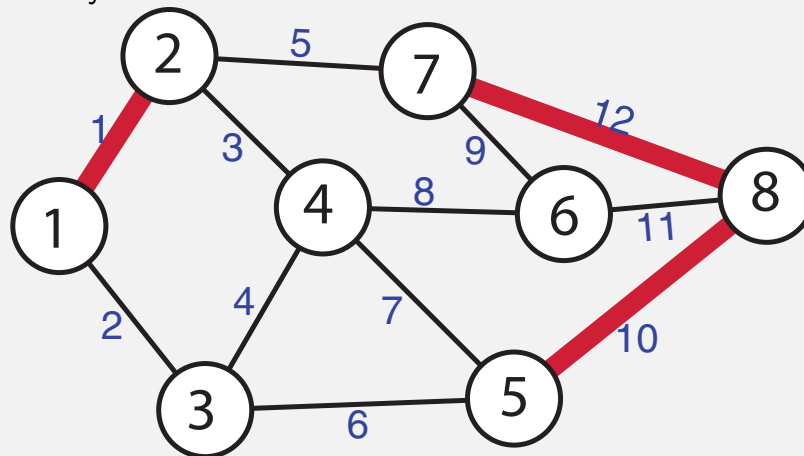
## Example: graphic matroid

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## Example: graphic matroid

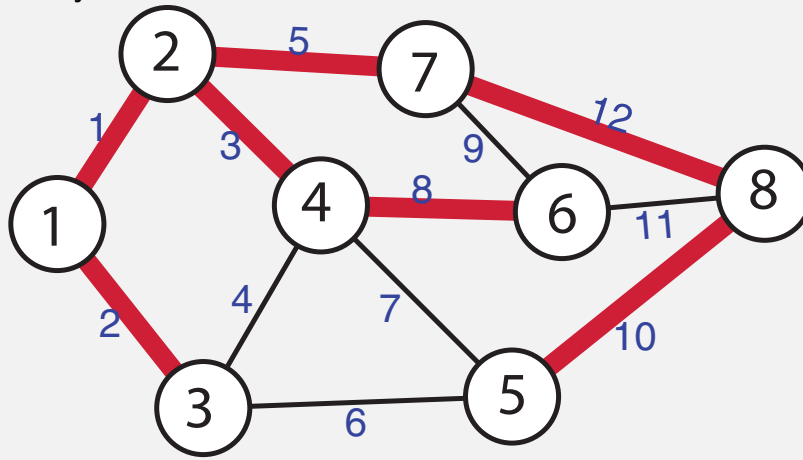
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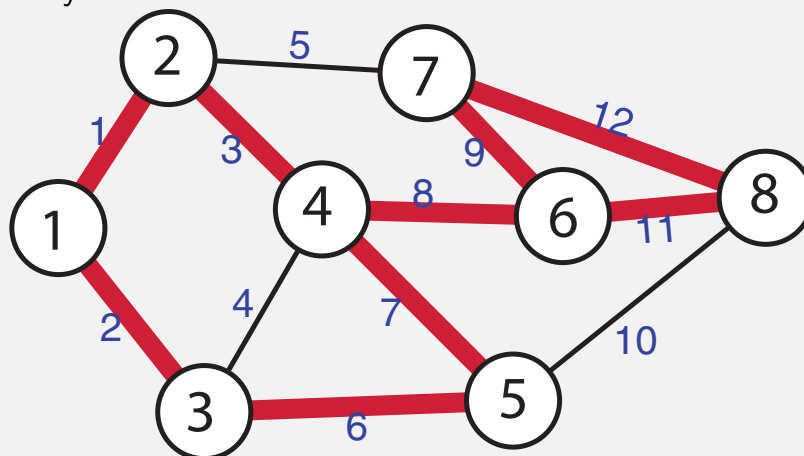
## Example: graphic matroid

- A graph defines a matroid on edge sets, independent sets are those without a cycle.



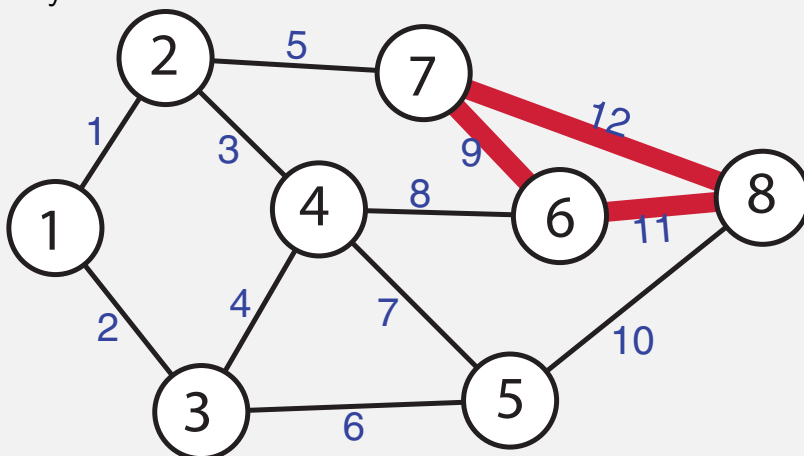
## Example: graphic matroid

- A graph defines a matroid on edge sets, independent sets are those without a cycle.



## Example: graphic matroid

- A graph defines a matroid on edge sets, independent sets are those without a cycle.



## Partition Matroid

- Let  $V$  be our ground set.
- Let  $V = V_1 \cup V_2 \cup \dots \cup V_\ell$  be a partition of  $V$  into  $\ell$  blocks (i.e., disjoint sets). Define a set of subsets of  $V$  as

$$\mathcal{I} = \{X \subseteq V : |X \cap V_i| \leq k_i \text{ for all } i = 1, \dots, \ell\}. \quad (6.10)$$

where  $k_1, \dots, k_\ell$  are fixed “limit” parameters,  $k_i \geq 0$ . Then  $M = (V, \mathcal{I})$  is a matroid.

- Note that a  $k$ -uniform matroid is a trivial example of a partition matroid with  $\ell = 1$ ,  $V_1 = V$ , and  $k_1 = k$ .
- Parameters associated with a partition matroid:  $\ell$  and  $k_1, k_2, \dots, k_\ell$  although often the  $k_i$ 's are all the same.
- We'll show that property (I3') in Def 6.3.4 holds. First note, for any  $X \subseteq V$ ,  $|X| = \sum_{i=1}^{\ell} |X \cap V_i|$  since  $\{V_1, V_2, \dots, V_\ell\}$  is a partition.
- If  $X, Y \in \mathcal{I}$  with  $|Y| > |X|$ , then there must be at least one  $i$  with  $|Y \cap V_i| > |X \cap V_i|$ . Therefore, adding one element  $e \in V_i \cap (Y \setminus X)$  to  $X$  won't break independence.

# Partition Matroid

Ground set of objects,  $V = \{$



# Partition Matroid

Partition of  $V$  into six blocks,  $V_1, V_2, \dots, V_6$



# Partition Matroid

Limit associated with each block,  $\{k_1, k_2, \dots, k_6\}$



# Partition Matroid

Independent subset but not maximally independent.



# Partition Matroid

Maximally independent subset, what is called a **base**.



# Partition Matroid

Not independent since over limit in set six.



## Matroids - rank function is submodular

### Lemma 6.5.1

The rank function  $r : 2^E \rightarrow \mathbb{Z}_+$  of a matroid is submodular, that is  $r(A) + r(B) \geq r(A \cup B) + r(A \cap B)$

### Proof.

- 1 Let  $X \in \mathcal{I}$  be an inclusionwise maximal set with  $X \subseteq A \cap B$
- 2 Let  $Y \in \mathcal{I}$  be inclusionwise maximal set with  $X \subseteq Y \subseteq A \cup B$ .
- 3 Since  $M$  is a matroid, we know that  $r(A \cap B) = r(X) = |X|$ , and  $r(A \cup B) = r(Y) = |Y|$ . Also, for any  $U \in \mathcal{I}$ ,  $r(A) \geq |A \cap U|$ .
- 4 Then we have (since  $X \subseteq A \cap B$ ,  $X \subseteq Y$ , and  $Y \subseteq A \cup B$ ),

$$r(A) + r(B) \geq |Y \cap A| + |Y \cap B| \quad (6.11)$$

$$= |Y \cap (A \cap B)| + |Y \cap (A \cup B)| \quad (6.12)$$

$$\geq |X| + |Y| = r(A \cap B) + r(A \cup B) \quad (6.13)$$

□

## A matroid is defined from its rank function

### Theorem 6.5.2 (Matroid from rank)

Let  $E$  be a set and let  $r : 2^E \rightarrow \mathbb{Z}_+$  be a function. Then  $r(\cdot)$  defines a matroid with  $r$  being its rank function if and only if for all  $A, B \subseteq E$ :

- (R1)  $\forall A \subseteq E \quad 0 \leq r(A) \leq |A|$  (non-negative cardinality bounded)
- (R2)  $r(A) \leq r(B)$  whenever  $A \subseteq B \subseteq E$  (monotone non-decreasing)
- (R3)  $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$  for all  $A, B \subseteq E$  (submodular)

- From above,  $r(\emptyset) = 0$ . Let  $v \notin A$ , then by monotonicity and submodularity,  $r(A) \leq r(A \cup \{v\}) \leq r(A) + r(\{v\})$  which gives only two possible values to  $r(A \cup \{v\})$ , namely  $r(A)$  or  $r(A) + 1$ .
- Hence, unit increment (if  $r(A) = k$ , then either  $r(A \cup \{v\}) = k$  or  $r(A \cup \{v\}) = k + 1$ ) holds.
- Thus, **submodularity**, **non-negative monotone non-decreasing**, and **unit increment** of rank is necessary and sufficient to define a matroid.
- Can refer to matroid as  $(E, r)$ ,  $E$  is ground set,  $r$  is rank function.

# Matroids from rank

## Proof of Theorem 6.5.2 (matroid from rank).

- Given a matroid  $M = (E, \mathcal{I})$ , we see its rank function as defined in Eq. 6.7 satisfies (R1), (R2), and, as we saw in Lemma 6.5.1, (R3) too.
- Next, assume we have (R1), (R2), and (R3). Define  $\mathcal{I} = \{X \subseteq E : r(X) = |X|\}$ . We will show that  $(E, \mathcal{I})$  is a matroid.
- First,  $\emptyset \in \mathcal{I}$ .
- Also, if  $Y \in \mathcal{I}$  and  $X \subseteq Y$  then by submodularity,

$$r(X) \geq r(Y) - r(Y \setminus X) + r(\emptyset) \quad (6.14)$$

$$\geq |Y| - |Y \setminus X| \quad (6.15)$$

$$= |X| \quad (6.16)$$

implying  $r(X) = |X|$ , and thus  $X \in \mathcal{I}$ .

...

# Matroids from rank

## Proof of Theorem 6.5.2 (matroid from rank) cont.

- Let  $A, B \in \mathcal{I}$ , with  $|A| < |B|$ , so  $r(A) = |A| < r(B) = |B|$ . Let  $B \setminus A = \{b_1, b_2, \dots, b_k\}$  (note  $1 \leq k \leq |B|$ ).
- Suppose, to the contrary, that  $\forall b \in B \setminus A, A + b \notin \mathcal{I}$ , which means for all such  $b$ ,  $r(A + b) = r(A) = |A| < |A| + 1$ . Then

$$r(B) \leq r(A \cup B) \quad (6.17)$$

$$\leq r(A \cup (B \setminus \{b_1\})) + r(A \cup \{b_1\}) - r(A) \quad (6.18)$$

$$= r(A \cup (B \setminus \{b_1\})) \quad (6.19)$$

$$\leq r(A \cup (B \setminus \{b_1, b_2\})) + r(A \cup \{b_2\}) - r(A) \quad (6.20)$$

$$= r(A \cup (B \setminus \{b_1, b_2\})) \quad (6.21)$$

$$\leq \dots \leq r(A) = |A| < |B| \quad (6.22)$$

giving a contradiction since  $B \in \mathcal{I}$ .

□

## Matroids from rank II

Another way of using function  $r$  to define a matroid.

### Theorem 6.5.3 (Matroid from rank II)

Let  $E$  be a finite set and let  $r : 2^E \rightarrow \mathbb{Z}_+$  be a function. Then  $r(\cdot)$  defines a matroid with  $r$  being its rank function if and only if for all  $X \subseteq E$ , and  $x, y \in E$ :

$$(R1') \quad r(\emptyset) = 0;$$

$$(R2') \quad r(X) \leq r(X \cup \{y\}) \leq r(X) + 1;$$

$$(R3') \quad \text{If } r(X \cup \{x\}) = r(X \cup \{y\}) = r(X), \text{ then } r(X \cup \{x, y\}) = r(X).$$

## Matroids by submodular functions

### Theorem 6.5.4 (Matroid by submodular functions)

Let  $f : 2^E \rightarrow \mathbb{Z}$  be a integer valued monotone non-decreasing submodular function. Define a set of sets as follows:

$$\mathcal{C}(f) = \left\{ C \subseteq E : \begin{array}{l} C \text{ is non-empty,} \\ \text{is inclusionwise-minimal,} \\ \text{and has } f(C) < |C| \end{array} \right\} \quad (6.23)$$

Then  $\mathcal{C}(f)$  is the collection of circuits of a matroid on  $E$ .

Inclusionwise-minimal in this case means that if  $C \in \mathcal{C}(f)$ , then there exists no  $C' \subset C$  with  $C' \in \mathcal{C}(f)$  (i.e.,  $C' \subset C$  would either be empty or have  $f(C') \geq |C'|$ ). Also, recall inclusionwise-minimal in Definition 6.3.10, the definition of a circuit.



## Summarizing: Many ways to define a Matroid

Summarizing what we've so far seen, we saw that it is possible to uniquely define a matroid based on any of:

- Independence (define the independent sets).
- Base axioms (exchangeability)
- Circuit axioms
- Closure axioms (we didn't see this, but it is possible)
- Rank axioms (normalized, monotone, cardinality bounded, non-negative integral, submodular)
- Matroids by integral submodular functions.

## Maximization problems for matroids

- Given a matroid  $M = (E, \mathcal{I})$  and a modular value function  $c : E \rightarrow \mathbb{R}$ , the task is to find an  $X \in \mathcal{I}$  such that  $c(X) = \sum_{x \in X} c(x)$  is maximum.
- This seems remarkably similar to the max spanning tree problem.

## Minimization problems for matroids

- Given a matroid  $M = (E, \mathcal{I})$  and a modular cost function  $c : E \rightarrow \mathbb{R}$ , the task is to find a basis  $B \in \mathcal{B}$  such that  $c(B)$  is minimized.
- This sounds like a set cover problem (find the minimum cost covering set of sets).

## Partition Matroid

- What is the partition matroid's rank function?
- A partition matroid's rank function:

$$r(A) = \sum_{i=1}^{\ell} \min(|A \cap V_i|, k_i) \quad (6.24)$$

which we also immediately see is submodular using properties we spoke about last week. That is:

- ①  $|A \cap V_i|$  is submodular (in fact modular) in  $A$
  - ②  $\min(\text{submodular}(A), k_i)$  is submodular in  $A$  since  $|A \cap V_i|$  is monotone.
  - ③ sums of submodular functions are submodular.
- $r(A)$  is also non-negative integral monotone non-decreasing, so it defines a matroid (the partition matroid).

## From 2-partition matroid rank to truncated matroid rank

- Example: 2-partition matroid rank function: Given natural numbers  $a, b \in \mathbb{Z}_+$  with  $a < b$ , and any set  $R \subseteq V$  with  $|R| = b$ .
- Create two-block partition  $V = (R, \bar{R})$ , where  $\bar{R} = V \setminus R$  so  $|\bar{R}| = |V| - b$ . Gives 2-partition matroid rank function as follows:

$$r(A) = \min(|A \cap R|, a) + \min(|A \cap \bar{R}|, |\bar{R}|) \quad (6.25)$$

$$= \min(|A \cap R|, a) + |A \cap \bar{R}| \quad (6.26)$$

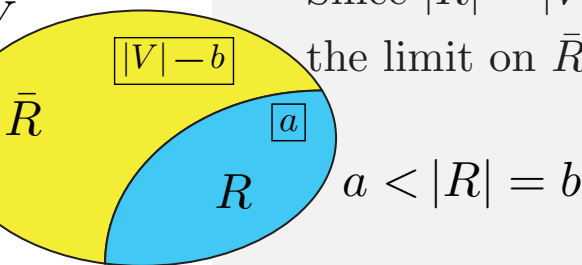
$$= \min(|A \cap \bar{R}| + |A \cap R|, |A \cap \bar{R}| + a) \quad (6.27)$$

$$= \min(|A|, |A \cap \bar{R}| + a) \quad (6.28)$$

- Figure showing partition blocks and partition matroid limits.

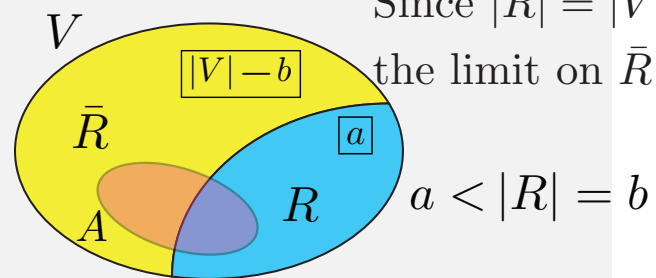
Since  $|\bar{R}| = |V| - b$

the limit on  $\bar{R}$  is vacuous.



Since  $|\bar{R}| = |V| - b$

the limit on  $\bar{R}$  is vacuous.



## Truncated Matroid Rank Function

- Define **truncated matroid rank** function. Start with 2-partition matroid rank  $r(A) = \min(|A \cap R|, a) + \min(|A \cap \bar{R}|, |\bar{R}|)$ ,  $a < b$ . Define:

$$f_R(A) = \min \{ r(A), b \} \quad (6.29)$$

$$= \min \{ \min(|A|, |A \cap \bar{R}| + a), b \} \quad (6.30)$$

$$= \min \{ |A|, a + |A \cap \bar{R}|, b \} \quad (6.31)$$

- Defines a matroid  $M = (V, f_R) = (V, \mathcal{I})$  (Goemans et. al.) with

$$\mathcal{I} = \{ I \subseteq V : |I| \leq b \text{ and } |I \cap R| \leq a \}, \quad (6.32)$$

Useful for showing hardness of constrained submodular minimization.

Consider sets  $B \subseteq V$  with  $|B| = b$ . Recall  $R$  fixed, and  $|R| = b$ .

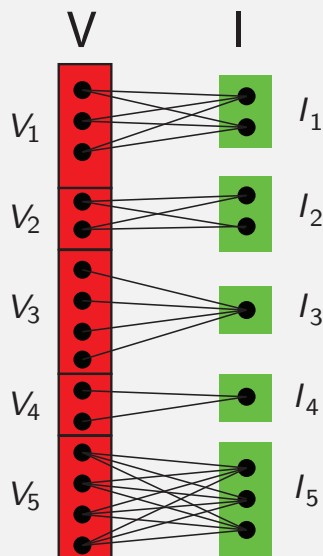
- For  $R$ , we have  $f_R(R) = \min(b, a, b) = a < b$ .
- For any  $B$  with  $|B \cap R| \leq a$ ,  $f_R(B) = b$ .
- For any  $B$  with  $|B \cap R| = \ell$ , with  $a \leq \ell \leq b$ ,  $f_R(B) = a + b - \ell$ .
- $R$ , the set with minimum valuation amongst size- $b$  sets, is hidden within an exponentially larger set of size- $b$  sets with larger valuation.

## Partition Matroid, rank as matching

- A partition matroid can be viewed using a bipartite graph.
- Letting  $V$  denote the ground set, and  $V_1, V_2, \dots$  the partition, the bipartite graph is  $G = (V, I, E)$  where  $V$  is the ground set,  $I$  is a set of “indices”, and  $E$  is the set of edges.
- $I = (I_1, I_2, \dots, I_\ell)$  is a set of  $k = \sum_{i=1}^{\ell} k_i$  nodes, grouped into  $\ell$  clusters, where there are  $k_i$  nodes in the  $i^{\text{th}}$  group  $I_i$ , and  $|I_i| = k_i$ .
- $(v, i) \in E(G)$  iff  $v \in V_j$  and  $i \in I_j$ .

## Partition Matroid, rank as matching

- Example where  $\ell = 5$ ,  
 $(k_1, k_2, k_3, k_4, k_5) =$   
 $(2, 2, 1, 1, 3)$ .



- Recall,  $\Gamma : 2^V \rightarrow \mathbb{R}$  as the neighbor function in a bipartite graph, the neighbors of  $X$  is defined as  $\Gamma(X) = \{v \in V(G) \setminus X : E(X, \{v\}) \neq \emptyset\}$ , and recall that  $|\Gamma(X)|$  is submodular.
- Here, for  $X \subseteq V$ , we have  $\Gamma(X) = \{i \in I : (v, i) \in E(G) \text{ and } v \in X\}$ .
- For such a constructed bipartite graph, the rank function of a partition matroid is  $r(X) = \sum_{i=1}^{\ell} \min(|X \cap V_i|, k_i)$  = the maximum matching involving  $X$ .