Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture 7 —

http://www.ee.washington.edu/people/faculty/bilmes/classes/ee563_spring_2018/

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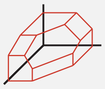
$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

$= f(A_r) + 2f(C) + f(B_r)$ $\quad = f(A_r) + f(C) + f(B_r)$ $\quad = f(A \cap B)$



Cumulative Outstanding Reading

- Read chapter 1 from Fujishige's book.
- Read chapter 2 from Fujishige's book.

Announcements, Assignments, and Reminders

- If you have any questions about anything, please ask then via our discussion board (`https://canvas.uw.edu/courses/1216339/discussion_topics`).

Class Road Map - EE563

- L1(3/26): Motivation, Applications, & Basic Definitions,
- L2(3/28): Machine Learning Apps (diversity, complexity, parameter, learning target, surrogate).
- L3(4/2): Info theory exs, more apps, definitions, graph/combinatorial examples
- L4(4/4): Graph and Combinatorial Examples, Matrix Rank, Examples and Properties, visualizations
- L5(4/9): More Examples/Properties/ Other Submodular Defs., Independence,
- L6(4/11): Matroids, Matroid Examples, Matroid Rank, Partition/Laminar Matroids
- L7(4/16): Laminar Matroids, System of Distinct Reps, Transversals, Transversal Matroid, Matroid Representation, Dual Matroids
- L8(4/18):
- L9(4/23):
- L10(4/25):
- L11(4/30):
- L12(5/2):
- L13(5/7):
- L14(5/9):
- L15(5/14):
- L16(5/16):
- L17(5/21):
- L18(5/23):
- L–(5/28): Memorial Day (holiday)
- L19(5/30):
- L21(6/4): Final Presentations maximization.

Last day of instruction, June 1st. Finals Week: June 2-8, 2018.

Matroid

Independent set definition of a matroid is perhaps most natural. Note, if $J \in \mathcal{I}$, then J is said to be an independent set.

Definition 7.2.3 (Matroid)

A set system $(E, \mathcal{I})$ is a Matroid if

(I1) $\emptyset \in \mathcal{I}$

(I2) $\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}$ (down-closed or subclusive)

(I3) $\forall I, J \in \mathcal{I}$, with $|I| = |J| + 1$, then there exists $x \in I \setminus J$ such that $J \cup \{x\} \in \mathcal{I}$.

Why is (I1) is not redundant given (I2)? Because without (I1) could have a non-matroid where $\mathcal{I} = \{\}$.

Matroids - important property

Proposition 7.2.3

In a matroid $M = (E, \mathcal{I})$, for any $U \subseteq E(M)$, any two bases of U have the same size.

- In matrix terms, given a set of vectors U, all sets of independent vectors that span the space spanned by U have the same size.
- In fact, under (I1),(I2), this condition is equivalent to (I3). Exercise: show the following is equivalent to the above.

Definition 7.2.4 (Matroid)

A set system $(V, \mathcal{I})$ is a Matroid if

(I1') $\emptyset \in \mathcal{I}$ (emptyset containing)

(I2') $\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}$ (down-closed or subclusive)

(I3') $\forall X \subseteq V$, and $I_1, I_2 \in \text{maxInd}(X)$, we have $|I_1| = |I_2|$ (all maximally independent subsets of X have the same size).

Partition Matroid

- Let V be our ground set.
- Let $V = V_1 \cup V_2 \cup \dots \cup V_\ell$ be a partition of V into ℓ blocks (i.e., disjoint sets). Define a set of subsets of V as

$$\mathcal{I} = \{X \subseteq V : |X \cap V_i| \leq k_i \text{ for all } i = 1, \dots, \ell\}. \tag{7.4}$$

  where $k_1, \dots, k_\ell$ are fixed "limit" parameters, $k_i \geq 0$. Then $M = (V, \mathcal{I})$ is a matroid.
- Note that a k-uniform matroid is a trivial example of a partition matroid with $\ell = 1$, $V_1 = V$, and $k_1 = k$.
- Parameters associated with a partition matroid: ℓ and $k_1, k_2, \dots, k_\ell$ although often the k_i's are all the same.
- We'll show that property (I3') in Def **??** holds. First note, for any $X \subseteq V$, $|X| = \sum_{i=1}^{\ell} |X \cap V_i|$ since $\{V_1, V_2, \dots, V_\ell\}$ is a partition.
- If $X, Y \in \mathcal{I}$ with $|Y| > |X|$, then there must be at least one i with $|Y \cap V_i| > |X \cap V_i|$. Therefore, adding one element $e \in V_i \cap (Y \setminus X)$ to X won't break independence.

Matroids - rank function is submodular

Lemma 7.2.3

The rank function $r : 2^E \to \mathbb{Z}_+$ of a matroid is submodular, that is $r(A) + r(B) \geq r(A \cup B) + r(A \cap B)$

Proof.

1. Let $X \in \mathcal{I}$ be an inclusionwise maximal set with $X \subseteq A \cap B$
2. Let $Y \in \mathcal{I}$ be inclusionwise maximal set with $X \subseteq Y \subseteq A \cup B$.
3. Since M is a matroid, we know that $r(A \cap B) = r(X) = |X|$, and $r(A \cup B) = r(Y) = |Y|$. Also, for any $U \in \mathcal{I}$, $r(A) \geq |A \cap U|$.
4. Then we have (since $X \subseteq A \cap B$, $X \subseteq Y$, and $Y \subseteq A \cup B$),

$$\begin{aligned} r(A) + r(B) &\geq |Y \cap A| + |Y \cap B| && (7.4) \\ &= |Y \cap (A \cap B)| + |Y \cap (A \cup B)| && (7.5) \\ &\geq |X| + |Y| = r(A \cap B) + r(A \cup B) && (7.6) \end{aligned}$$

$\square$

A matroid is defined from its rank function

Theorem 7.2.3 (Matroid from rank)

Let E be a set and let $r : 2^E \to \mathbb{Z}_+$ be a function. Then $r(\cdot)$ defines a matroid with r being its rank function if and only if for all $A, B \subseteq E$:

(R1) $\forall A \subseteq E \;\; 0 \leq r(A) \leq |A|$ *(non-negative cardinality bounded)*

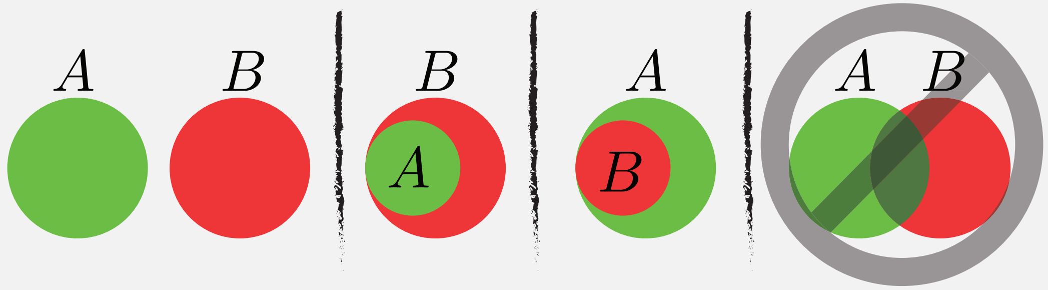
(R2) $r(A) \leq r(B)$ *whenever* $A \subseteq B \subseteq E$ *(monotone non-decreasing)*

(R3) $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$ *for all* $A, B \subseteq E$ *(submodular)*

- From above, $r(\emptyset) = 0$. Let $v \notin A$, then by monotonicity and submodularity, $r(A) \leq r(A \cup \{v\}) \leq r(A) + r(\{v\})$ which gives only two possible values to $r(A \cup \{v\})$, namely $r(A)$ or $r(A) + 1$.
- Hence, unit increment (if $r(A) = k$, then either $r(A \cup \{v\}) = k$ or $r(A \cup \{v\}) = k + 1$) holds.
- Thus, submodularity, non-negative monotone non-decreasing, and unit increment of rank is necessary and sufficient to define a matroid.
- Can refer to matroid as (E, r), E is ground set, r is rank function.

Laminar Family and Laminar Matroid

- We can define a matroid with structures richer than just partitions.
- A set system $(V, \mathcal{F})$ is called a laminar family if for any two sets $A, B \in \mathcal{F}$, at least one of the three sets $A \cap B$, $A \setminus B$, or $B \setminus A$ is empty.



- Family is laminar $\exists$ no two properly intersecting members: $\forall A, B \in \mathcal{F}$, either A, B disjoint ($A \cap B = \emptyset$) or comparable ($A \subseteq B$ or $B \subseteq A$).
- Suppose we have a laminar family $\mathcal{F}$ of subsets of V and an integer k_A for every set $A \in \mathcal{F}$. Then $(V, \mathcal{I})$ defines a matroid where

$$\mathcal{I} = \{I \subseteq E : |I \cap A| \leq k_A \text{ for all } A \in \mathcal{F}\} \tag{7.1}$$

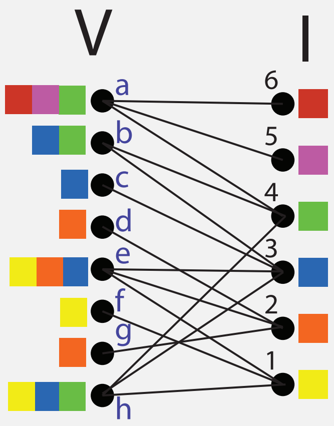
- Exercise: what is the rank function here?

System of Representatives

- Let $(V, \mathcal{V})$ be a set system (i.e., $\mathcal{V} = (V_i : i \in I)$ where $\emptyset \subset V_i \subseteq V$ for all i), and I is an index set. Hence, $|I| = |\mathcal{V}|$.
- Here, the sets $V_i \in \mathcal{V}$ are like "groups" and any $v \in V$ with $v \in V_i$ is a member of group i. Groups need not be disjoint (e.g., interest groups of individuals).
- A family $(v_i : i \in I)$ with $v_i \in V$ is said to be a system of representatives of $\mathcal{V}$ if $\exists$ a bijection $\pi : I \to I$ such that $v_i \in V_{\pi(i)}$.
- v_i is the representative of set (or group) $V_{\pi(i)}$, meaning the i^{th} representative is meant to represent set $V_{\pi(i)}$.
- Example: Consider the house of representatives, $v_i =$ "Jim McDermott", while $i =$ "King County, WA-7".
- In a system of representatives, there is no requirement for the representatives to be distinct. I.e., we could have some $v_1 \in V_1 \cap V_2$, where v_1 represents both V_1 and V_2.
- We can view this as a bipartite graph.

System of Representatives

- We can view this as a bipartite graph. The groups of V are marked by color tags on the left, and also via right neighbors in the graph.
- Here, $\ell = 6$ groups, with $\mathcal{V} = (V_1, V_2, \ldots, V_6)$
$= \Big(\{e, f, h\}, \{d, e, g\}, \{b, c, e, h\}, \{a, b, h\}, \{a\}, \{a\}\Big).$



- A system of representatives would make sure that there is a representative for each color group. For example,
- The representatives $(\{a, c, d, f, h\})$ are shown as colors on the left.
- Here, the set of representatives is not distinct. Why? In fact, due to the red and pink group, a distinct group of representatives is impossible (since there is only one common choice to represent both color groups).

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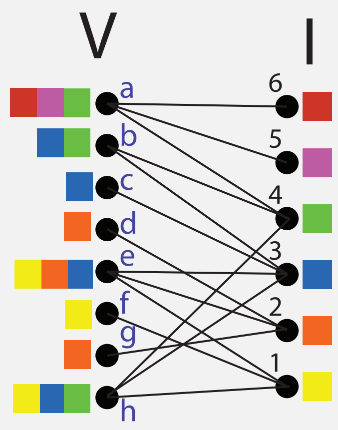

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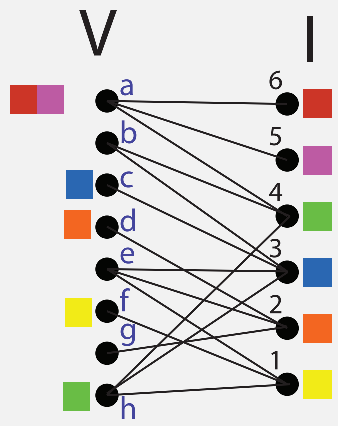

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System of Distinct Representatives

- Let $(V, \mathcal{V})$ be a set system (i.e., $\mathcal{V} = (V_k : i \in I)$ where $V_i \subseteq V$ for all i), and I is an index set. Hence, $|I| = |\mathcal{V}|$.
- A family $(v_i : i \in I)$ with $v_i \in V$ is said to be a system of distinct representatives of $\mathcal{V}$ if $\exists$ a bijection $\pi : I \leftrightarrow I$ such that $v_i \in V_{\pi(i)}$ and $v_i \neq v_j$ for all $i \neq j$.
- In a system of distinct representatives, there is a requirement for the representatives to be distinct. We can re-state (and rename) this as a:

Definition 7.4.1 (transversal)

Given a set system $(V, \mathcal{V})$ and index set I for $\mathcal{V}$ as defined above, a set $T \subseteq V$ is a transversal of $\mathcal{V}$ if there is a bijection $\pi : T \leftrightarrow I$ such that

$$x \in V_{\pi(x)} \text{ for all } x \in T \tag{7.2}$$

- Note that due to $\pi : T \leftrightarrow I$ being a bijection, all of I and T are "covered" (so this makes things distinct automatically).

Transversals are Subclusive

- A set $T' \subseteq V$ is a partial transversal if T' is a transversal of some subfamily $\mathcal{V}' = (V_i : i \in I')$ where $I' \subseteq I$.
- Therefore, for any transversal T, any subset $T' \subseteq T$ is a partial transversal.
- Thus, transversals are down closed (subclusive).

When do transversals exist?

- As we saw, a transversal might not always exist. How to tell?
- Given a set system $(V, \mathcal{V})$ with $\mathcal{V} = (V_i : i \in I)$, and $V_i \subseteq V$ for all i. Then, for any $J \subseteq I$, let

$$V(J) = \cup_{j \in J} V_j \tag{7.3}$$

so $|V(J)| : 2^I \to \mathbb{Z}_+$ is the set cover func. (we know is submodular).
- We have

Theorem 7.5.1 (Hall's theorem)

Given a set system $(V, \mathcal{V})$*, the family of subsets* $\mathcal{V} = (V_i : i \in I)$ *has a transversal* $(v_i : i \in I)$ *iff for all* $J \subseteq I$

$$|V(J)| \geq |J| \tag{7.4}$$

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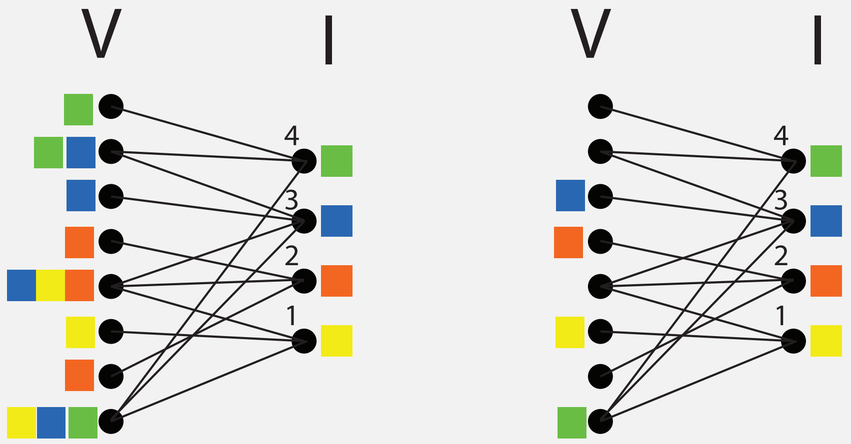
so $|V(J)| : 2^I \to \mathbb{Z}_+$ is the set cover func. (we know is submodular).
- Hall's theorem $(\forall J \subseteq I, |V(J)| \geq |J|)$ as a bipartite graph.



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$$V(J) = \cup_{j \in J} V_j \tag{7.3}$$

so $|V(J)| : 2^I \to \mathbb{Z}_+$ is the set cover func. (we know is submodular).
- Moreover, we have

Theorem 7.5.2 (Rado's theorem (1942))

If $M = (V, r)$ is a matroid on V with rank function r, then the family of subsets $(V_i : i \in I)$ of V has a transversal $(v_i : i \in I)$ that is independent in M iff for all $J \subseteq I$

$$r(V(J)) \geq |J| \tag{7.5}$$

- Note, a transversal T independent in M means that $r(T) = |T|$.

More general conditions for existence of transversals

Theorem 7.5.3 (Polymatroid transversal theorem)

If $\mathcal{V} = (V_i : i \in I)$ is a finite family of non-empty subsets of V, and $f : 2^V \to \mathbb{Z}_+$ is a non-negative, integral, monotone non-decreasing, and submodular function, then $\mathcal{V}$ has a system of representatives $(v_i : i \in I)$ such that

$$f(\cup_{i \in J}\{v_i\}) \geq |J| \text{ for all } J \subseteq I \tag{7.6}$$

if and only if

$$f(V(J)) \geq |J| \text{ for all } J \subseteq I \tag{7.7}$$

- Given Theorem 7.5.3, we immediately get Theorem 7.5.1 by taking $f(S) = |S|$ for $S \subseteq V$. *In which case, Eq. 7.6 requires the system of representatives to be distinct.*
- We get Theorem 7.5.2 by taking $f(S) = r(S)$ for $S \subseteq V$, the rank function of the matroid. *where, Eq. 7.6 insists the system of representatives is independent in M, and hence also distinct.*

Submodular Composition with Set-to-Set functions

- Note the condition in Theorem 7.5.3 is $f(V(J)) \geq |J|$ for all $J \subseteq I$, where $f : 2^V \to \mathbb{Z}_+$ is non-negative, integral, monotone non-decreasing and submodular, and $V(J) = \cup_{j \in J} V_j$ with $V_i \subseteq V$.
- Note $V(\cdot) : 2^I \to 2^V$ is a set-to-set function, composable with a submodular function.
- Define $g : 2^I \to \mathbb{Z}$ with $g(J) = f(V(J)) - |J|$, then the condition for the existence of a system of representatives, with quality Equation 7.6, becomes:

$$\min_{J \subseteq I} g(J) \geq 0 \tag{7.8}$$

- What kind of function is g?

Proposition 7.5.4

g as given above is submodular.

- Hence, the condition for existence can be solved by (a special case of) submodular function minimization, or vice verse!

More general conditions for existence of transversals

first part proof of Theorem 7.5.3.

- Suppose $\mathcal{V}$ has a system of representatives $(v_i : i \in I)$ such that Eq. 7.6 (i.e., $f(\cup_{i \in J}\{v_i\}) \geq |J|$ for all $J \subseteq I$) is true.
- Then since f is monotone, and since $V(J) \supseteq \cup_{i \in J}\{v_i\}$ when $(v_i : i \in I)$ is a system of representatives, then Eq. 7.7 (i.e., $f(V(J)) \geq |J|$ for all $J \subseteq I$) immediately follows.

...

More general conditions for existence of transversals

Lemma 7.5.5 (contraction lemma)

Suppose Eq. 7.7 ($f(V(J)) \geq |J|, \forall J \subseteq I$) is true for $\mathcal{V} = (V_i : i \in I)$, and there exists an i such that $|V_i| \geq 2$ (w.l.o.g., say $i = 1$). Then there exists $\bar{v} \in V_1$ such that the family of subsets $(V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_{|I|})$ also satisfies Eq 7.7.

Proof.

- When Eq. 7.7 holds, this means that for any subsets $J_1, J_2 \subseteq I \setminus \{1\}$, we have that, for $J \in \{J_1, J_2\}$,

$$f(V(J \cup \{1\})) \geq |J \cup \{1\}| \tag{7.9}$$

and hence

$$f(V_1 \cup V(J_1)) \geq |J_1| + 1 \tag{7.10}$$

$$f(V_1 \cup V(J_2)) \geq |J_2| + 1 \tag{7.11}$$

...

More general conditions for existence of transversals

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Proof.

- Suppose, to the contrary, the consequent is false. Then we may take any $\bar{v}_1, \bar{v}_2 \in V_1$ as two distinct elements in V_1 ...
- ...and there must exist subsets J_1, J_2 of $I \setminus \{1\}$ such that

$$f((V_1 \setminus \{\bar{v}_1\}) \cup V(J_1)) < |J_1| + 1, \tag{7.12}$$

$$f((V_1 \setminus \{\bar{v}_2\}) \cup V(J_2)) < |J_2| + 1, \tag{7.13}$$

(note that either one or both of J_1, J_2 could be empty).

...

More general conditions for existence of transversals

Lemma 7.5.5 (contraction lemma)

Suppose Eq. 7.7 ($f(V(J)) \geq |J|, \forall J \subseteq I$) is true for $\mathcal{V} = (V_i : i \in I)$, and there exists an i such that $|V_i| \geq 2$ (w.l.o.g., say $i = 1$). Then there exists $\bar{v} \in V_1$ such that the family of subsets $(V_1 \setminus \{\bar{v}\}, V_2, \dots, V_{|I|})$ also satisfies Eq 7.7.

Proof.

- Taking $X = (V_1 \setminus \{\bar{v}_1\}) \cup V(J_1)$ and $Y = (V_1 \setminus \{\bar{v}_2\}) \cup V(J_2)$, we have $f(X) \leq |J_1|$, $f(Y) \leq |J_2|$, and that:

  $$X \cup Y = V_1 \cup V(J_1 \cup J_2), \tag{7.14}$$

  $$X \cap Y \supseteq V(J_1 \cap J_2), \tag{7.15}$$

  and

  $$\begin{aligned} |J_1| + |J_2| &\geq f(X) + f(Y) \\ &\geq f(X \cup Y) + f(X \cap Y) \end{aligned} \tag{7.16}$$

...

More general conditions for existence of transversals

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Proof.

- since f submodular monotone non-decreasing, & Eqs 7.14-7.16,

  $$|J_1| + |J_2| \geq f(V_1 \cup V(J_1 \cup J_2)) + f(V(J_1 \cap J_2)) \tag{7.17}$$

- Since $\mathcal{V}$ satisfies Eq. 7.7, $1 \notin J_1 \cup J_2$, & Eqs 7.10-7.11, this gives

  $$|J_1| + |J_2| \geq |J_1 \cup J_2| + 1 + |J_1 \cap J_2| \tag{7.18}$$

  which is a contradiction since cardinality is modular.

...

More general conditions for existence of transversals

Theorem 7.5.3 (Polymatroid transversal theorem)

If $\mathcal{V} = (V_i : i \in I)$ *is a finite family of non-empty subsets of* V*, and* $f : 2^V \to \mathbb{Z}_+$ *is a non-negative, integral, monotone non-decreasing, and submodular function, then* $\mathcal{V}$ *has a system of representatives* $(v_i : i \in I)$ *such that*

$$f(\cup_{i \in J}\{v_i\}) \geq |J| \text{ for all } J \subseteq I \tag{7.6}$$

if and only if

$$f(V(J)) \geq |J| \text{ for all } J \subseteq I \tag{7.7}$$

- Given Theorem 7.5.3, we immediately get Theorem 7.5.1 by taking $f(S) = |S|$ for $S \subseteq V$. *In which case, Eq. 7.6 requires the system of representatives to be distinct.*
- We get Theorem 7.5.2 by taking $f(S) = r(S)$ for $S \subseteq V$, the rank function of the matroid. *where, Eq. 7.6 insists the system of representatives is independent in* M*, and hence also distinct.*

More general conditions for existence of transversals

converse proof of Theorem 7.5.3.

- Conversely, suppose Eq. 7.7 is true.
- If each V_i is a singleton set, then the result follows immediately.
- W.l.o.g., let $|V_1| \geq 2$, then by Lemma 7.5.5, the family of subsets $(V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_{|I|})$ also satisfies Eq 7.7 for the right $\bar{v}$.
- We can continue to reduce the family, deleting elements from V_i for some i while $|V_i| \geq 2$, until we arrive at a family of singleton sets.
- This family will be the required system of representatives.

☐

This theorem can be used to produce a variety of other results quite easily, and shows how submodularity is the key ingredient in its truth.

Transversal Matroid

Transversals, themselves, define a matroid.

Theorem 7.6.1

If $\mathcal{V}$ is a family of finite subsets of a ground set V, then the collection of partial transversals of $\mathcal{V}$ is the set of independent sets of a matroid $M = (V, \mathcal{V})$ on V.

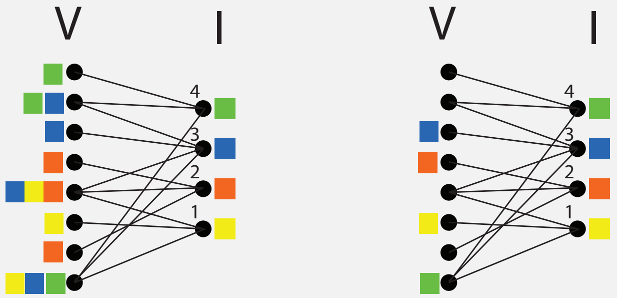
- This means that the transversals of $\mathcal{V}$ are the bases of matroid M.
- Therefore, all maximal partial transversals of $\mathcal{V}$ have the same cardinality!

Transversals and Bipartite Matchings

- Transversals correspond exactly to matchings in bipartite graphs.
- Given a set system $(V, \mathcal{V})$, with $\mathcal{V} = (V_i : i \in I)$, we can define a bipartite graph $G = (V, I, E)$ associated with $\mathcal{V}$ that has edge set $\{(v, i) : v \in V, i \in I, v \in V_i\}$.
- A matching in this graph is a set of edges no two of which that have a common endpoint. In fact, we easily have:

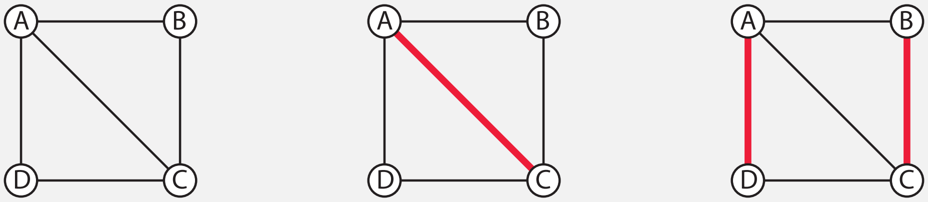
Lemma 7.6.2

A subset $T \subseteq V$ is a partial transversal of $\mathcal{V}$ iff there is a matching in (V, I, E) in which every edge has one endpoint in T (T matched into I).



Arbitrary Matchings and Matroids?

- Are arbitrary matchings matroids?
- Consider the following graph (left), and two max-matchings (two right instances)



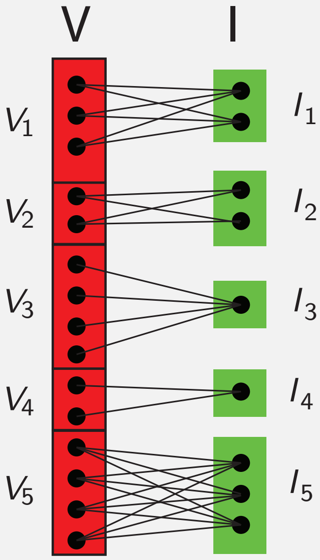
- $\{AC\}$ is a maximum matching, as is $\{AD, BC\}$, but they are not the same size.
- Let $\mathcal{M}$ be the set of matchings in an arbitrary graph $G = (V, E)$. Hence, $(E, \mathcal{M})$ is a set system. I1 holds since $\emptyset \in \mathcal{M}$. I2 also holds since if $M \in \mathcal{M}$ is a matching, then so is any $M' \subseteq M$. I3 doesn't hold (as seen above). Exercise: fully characterize the problem of finding the largest subset $\mathcal{M}' \subset \mathcal{M}$ of matchings so that $(E, \mathcal{M}')$ also satisfies I3?

Review from Lecture 7

The next frame comes from lecture 7.

Partition Matroid, rank as matching

- Example where $\ell = 5$, $(k_1, k_2, k_3, k_4, k_5) = (2, 2, 1, 1, 3)$.



- Recall, $\Gamma : 2^V \to \mathbb{R}$ as the neighbor function in a bipartite graph, the neighbors of X is defined as $\Gamma(X) = \{v \in V(G) \setminus X : E(X, \{v\}) \neq \emptyset\}$, and recall that $|\Gamma(X)|$ is submodular.
- Here, for $X \subseteq V$, we have $\Gamma(X) = \{i \in I : (v, i) \in E(G) \text{ and } v \in X\}$.
- For such a constructed bipartite graph, the rank function of a partition matroid is $r(X) = \sum_{i=1}^{\ell} \min(|X \cap V_i|, k_i) =$ the maximum matching involving X.

Morphing Partition Matroid Rank

- Recall the partition matroid rank function. Note, $k_i = |I_i|$ in the bipartite graph representation, and since a matroid, w.l.o.g., $|V_i| \geq k_i$ (also, recall, $V(J) = \cup_{j \in J} V_j$).
- Start with partition matroid rank function in the subsequent equations.

$$r(A) = \sum_{i \in \{1,\dots,\ell\}} \min(|A \cap V_i|, k_i) \tag{7.19}$$

$$= \sum_{i=1}^{\ell} \min(|A \cap V(I_i)|, |I_i|) \tag{7.20}$$

$$= \sum_{i \in \{1,\dots,\ell\}} \min_{J_i \in \{\emptyset, I_i\}} \left(\begin{Bmatrix} |A \cap V(I_i)| & \text{if } J_i \neq \emptyset \\ 0 & \text{if } J_i = \emptyset \end{Bmatrix} + |I_i \setminus J_i| \right) \tag{7.21}$$

$$= \sum_{i \in \{1,\dots,\ell\}} \min_{J_i \subseteq I_i} \left(\begin{Bmatrix} |A \cap V(I_i)| & \text{if } J_i \neq \emptyset \\ 0 & \text{if } J_i = \emptyset \end{Bmatrix} + |I_i \setminus J_i| \right) \tag{7.22}$$

$$= \sum_{i \in \{1,\dots,\ell\}} \min_{J_i \subseteq I_i} \left(|V(J_i) \cap A| + |I_i \setminus J_i| \right) \tag{7.23}$$

… Morphing Partition Matroid Rank

- Continuing,

$$r(A) = \sum_{i=1}^{\ell} \min_{J_i \subseteq I_i} \left(|V(J_i) \cap V(I_i) \cap A| - |I_i \cap J_i| + |I_i|\right) \tag{7.24}$$

$$= \min_{J \subseteq I} \left(\sum_{i=1}^{\ell} |V(J) \cap V(I_i) \cap A| - |I_i \cap J| + |I_i| \right) \tag{7.25}$$

$$= \min_{J \subseteq I} \left(|V(J) \cap V(I) \cap A| - |J| + |I|\right) \tag{7.26}$$

$$= \min_{J \subseteq I} \left(|V(J) \cap A| - |J| + |I|\right) \tag{7.27}$$

- In fact, this bottom (more general) expression is the expression for the rank of a transversal matroid.

Partial Transversals Are Independent Sets in a Matroid

In fact, we have

Theorem 7.6.3

Let $(V, \mathcal{V})$ where $\mathcal{V} = (V_1, V_2, \dots, V_\ell)$ be a subset system. Let $I = \{1, \dots, \ell\}$. Let $\mathcal{I}$ be the set of partial transversals of $\mathcal{V}$. Then $(V, \mathcal{I})$ is a matroid.

Proof.

- We note that $\emptyset \in \mathcal{I}$ since the empty set is a transversal of the empty subfamily of $\mathcal{V}$, thus (I1') holds.
- We already saw that if T is a partial transversal of $\mathcal{V}$, and if $T' \subseteq T$, then T' is also a partial transversal. So (I2') holds.
- Suppose that T_1 and T_2 are partial transversals of $\mathcal{V}$ such that $|T_1| < |T_2|$. Exercise: show that (I3') holds.

□

Transversal Matroid Rank

- Transversal matroid has rank

$$r(A) = \min_{J \subseteq I} \left(|V(J) \cap A| - |J| + |I| \right) \tag{7.28}$$

$$= \min_{J \subseteq I} m_J(I) \tag{7.29}$$

- Therefore, this function is submodular.
- Note that it is a minimum over a set of modular functions in I. Is this true in general? Exercise:
- Exercise: Can you identify a set of sufficient properties over a set of modular functions $m_i : V \to \mathbb{R}_+$ so that $f(A) = \min_i m_i(A)$ is submodular? Can you identify both necessary and sufficient conditions?

Matroid loops

- A circuit in a matroids is well defined, a subset $A \subseteq E$ is circuit if it is an inclusionwise minimally dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).
- There is no reason in a matroid such an A could not consist of a single element.
- Such an $\{a\}$ is called a loop.
- In a matric (i.e., linear) matroid, the only such loop is the value $\mathbf{0}$, as all non-zero vectors have rank 1. The $\mathbf{0}$ can appear > 1 time with different indices, as can a self loop in a graph appear on different nodes.
- Note, we also say that two elements s, t are said to be parallel if $\{s, t\}$ is a circuit.

Representable

Definition 7.7.1 (Matroid isomorphism)

Two matroids M_1 and M_2 respectively on ground sets V_1 and V_2 are isomorphic if there is a bijection $\pi : V_1 \to V_2$ which preserves independence (equivalently, rank, circuits, and so on).

- Let $\mathbb{F}$ be any field (such as $\mathbb{R}$, $\mathbb{Q}$, or some finite field $\mathbb{F}$, such as a Galois field GF(p) where p is prime (such as GF(2)), but not $\mathbb{Z}$. Succinctly: A field is a set with $+$, $*$, closure, associativity, commutativity, and additive and multiplicative identities and inverses.
- We can more generally define matroids on a field.

Definition 7.7.2 (linear matroids on a field)

Let $\mathbf{X}$ be an $n \times m$ matrix and $E = \{1, \dots, m\}$, where $\mathbf{X}_{ij} \in \mathbb{F}$ for some field, and let $\mathcal{I}$ be the set of subsets of E such that the columns of $\mathbf{X}$ are linearly independent over $\mathbb{F}$.

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- We can more generally define matroids on a field.

Definition 7.7.3 (representable (as a linear matroid))

Any matroid isomorphic to a linear matroid on a field is called representable over $\mathbb{F}$

Representability of Transversal Matroids

- Piff and Welsh in 1970, and Adkin in 1972 proved an important theorem about representability of transversal matroids.
- In particular:

Theorem 7.7.4

Transversal matroids are representable over all finite fields of sufficiently large cardinality, and are representable over any infinite field.

Converse: Representability of Transversal Matroids

The converse is not true, however.

Example 7.7.5

Let $V = \{1, 2, 3, 4, 5, 6\}$ be a ground set and let $M = (V, \mathcal{I})$ be a set system where $\mathcal{I}$ is all subsets of V of cardinality ≤ 2 except for the pairs $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$.

- It can be shown that this is a matroid and is representable.
- However, this matroid is not isomorphic to any transversal matroid.

Review from Lecture 6

The next frame comes from lecture 6.

Matroids, other definitions using matroid rank $r : 2^V \to \mathbb{Z}_+$

Definition 7.8.3 (closed/flat/subspace)

A subset $A \subseteq E$ is closed (equivalently, a flat or a subspace) of matroid M if for all $x \in E \setminus A$, $r(A \cup \{x\}) = r(A) + 1$.

Definition: A hyperplane is a flat of rank $r(M) - 1$.

Definition 7.8.4 (closure)

Given $A \subseteq E$, the closure (or span) of A, is defined by $\text{span}(A) = \{b \in E : r(A \cup \{b\}) = r(A)\}$.

Therefore, a closed set A has $\text{span}(A) = A$.

Definition 7.8.5 (circuit)

A subset $A \subseteq E$ is circuit or a cycle if it is an inclusionwise-minimal dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).

Spanning Sets

- We have the following definitions:

Definition 7.8.1 (spanning set of a set)

Given a matroid $\mathcal{M} = (V, \mathcal{I})$, and a set $Y \subseteq V$, then any set $X \subseteq Y$ such that $r(X) = r(Y)$ is called a spanning set of Y.

Definition 7.8.2 (spanning set of a matroid)

Given a matroid $\mathcal{M} = (V, \mathcal{I})$, any set $A \subseteq V$ such that $r(A) = r(V)$ is called a spanning set of the matroid.

- A base of a matroid is a minimal spanning set (and it is independent) but supersets of a base are also spanning.
- V is always trivially spanning.
- Consider the terminology: "spanning tree in a graph", comes from spanning in a matroid sense.

Dual of a Matroid

- Given a matroid $M = (V, \mathcal{I})$, a dual matroid $M^* = (V, \mathcal{I}^*)$ can be defined on the same ground set V, but using a very different set of independent sets $\mathcal{I}^*$.
- We define the set of sets $\mathcal{I}^*$ for M^* as follows:

$$\begin{aligned} \mathcal{I}^* &= \{A \subseteq V : V \setminus A \text{ is a spanning set of } M\} && (7.30) \\ &= \{V \setminus S : S \subseteq V \text{ is a spanning set of } M\} && (7.31) \end{aligned}$$

  i.e., $\mathcal{I}^*$ are complements of spanning sets of M.
- That is, a set A is independent in the dual matroid M^* if removal of A from V does not decrease the rank in M:

$$\mathcal{I}^* = \{A \subseteq V : \text{rank}_M(V \setminus A) = \text{rank}_M(V)\} \qquad (7.32)$$

- In other words, a set $A \subseteq V$ is independent in the dual M^* (i.e., $A \in \mathcal{I}^*$) if A's complement is spanning in M (residual $V \setminus A$ must contain a base in M).
- Dual of the dual: Note, we have that $(M^*)^* = M$.

Dual of a Matroid: Bases

- The smallest spanning sets are bases. Hence, a base B of M (where $B = V \setminus B^*$ is as small as possible while still spanning) is the complement of a base B^* of M^* (where $B^* = V \setminus B$ is as large as possible while still being independent).
- In fact, we have that

Theorem 7.8.3 (Dual matroid bases)

Let $M = (V, \mathcal{I})$ be a matroid and $\mathcal{B}(M)$ be the set of bases of M. Then define

$$\mathcal{B}^*(M) = \{V \setminus B : B \in \mathcal{B}(M)\}. \tag{7.33}$$

Then $\mathcal{B}^*(M)$ is the set of basis of M^* (that is, $\mathcal{B}^*(M) = \mathcal{B}(M^*)$.

An exercise in duality Terminology

- $\mathcal{B}^*(M)$, the bases of M^*, are called cobases of M.
- The circuits of M^* are called cocircuits of M.
- The hyperplanes of M^* are called cohyperplanes of M.
- The independent sets of M^* are called coindependent sets of M.
- The spanning sets of M^* are called cospanning sets of M.

Proposition 7.8.4 (from Oxley 2011)

Let $M = (V, \mathcal{I})$ be a matroid, and let $X \subseteq V$. Then

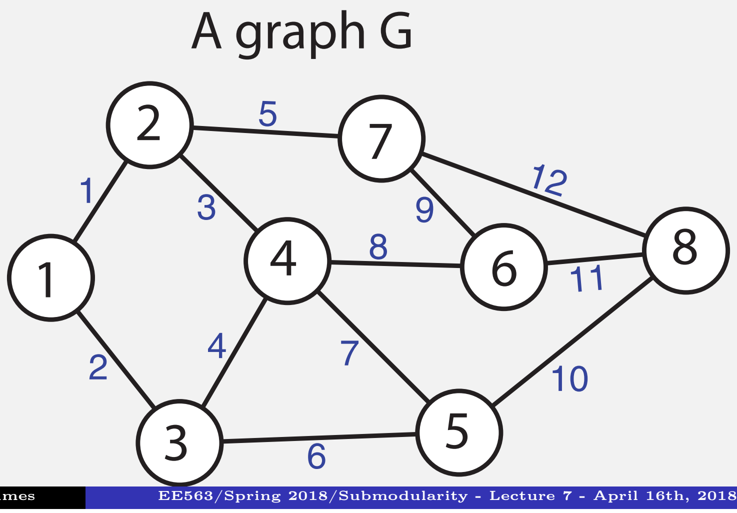
1. *X is independent in M iff $V \setminus X$ is cospanning in M (spanning in M^*).*
2. *X is spanning in M iff $V \setminus X$ is coindependent in M (independent in M^*).*
3. *X is a hyperplane in M iff $V \setminus X$ is a cocircuit in M (circuit in M^*).*
4. *X is a circuit in M iff $V \setminus X$ is a cohyperplane in M (hyperplane in M^*).*

Example duality: graphic matroid

- Using a graphic/cycle matroid, we can already see how dual matroid concepts demonstrates the extraordinary flexibility and power that a matroid can have.
- Recall, in cycle matroid, a spanning set of G is any set of edges that are incident to all nodes (i.e., any superset of a spanning forest), a minimal spanning set is a spanning tree (or forest), and a circuit has a nice visual interpretation (a cycle in the graph).
- A cut in a graph G is a set of edges, the removal of which increases the number of connected components. I.e., $X \subseteq E(G)$ is a cut in G if $k(G) < k(G \setminus X)$.
- A minimal cut in G is a cut $X \subseteq E(G)$ such that $X \setminus \{x\}$ is not a cut for any $x \in X$.
- A cocycle (cocircuit) in a graphic matroid is a minimal graph cut.
- A mincut is a circuit in the dual "cycle" (or "cut") matroid.
- All dependent sets in a cocycle matroid are cuts (i.e., a dependent set is a minimal cut or contains one).

Example: cocycle matroid (sometimes "cut matroid")

- The dual of the cycle matroid is called the cocycle matroid. Recall, $\mathcal{I}^* = \{A \subseteq V : V \setminus A \text{ is a spanning set of } M\}$
- $\mathcal{I}^*$ consists of all sets of edges the complement of which contains a spanning tree — i.e., an independent set can't consist of edges that, if removed, would render the graph non-spanning.



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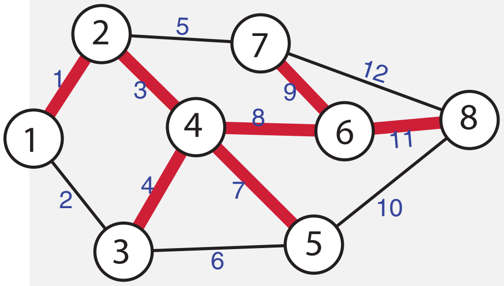
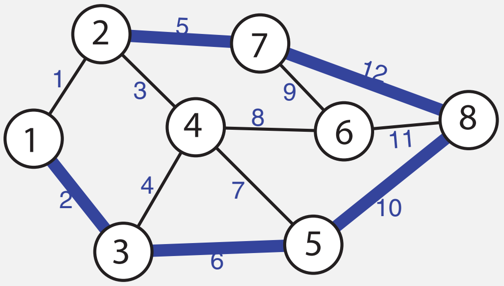
Minimally spanning in M (and thus a base (maximally independent) in M)



Maximally independent in M* (thus a base, minimally spanning, in M*)



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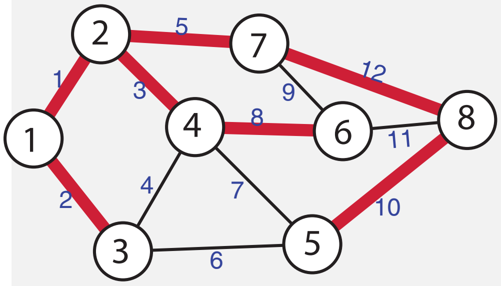

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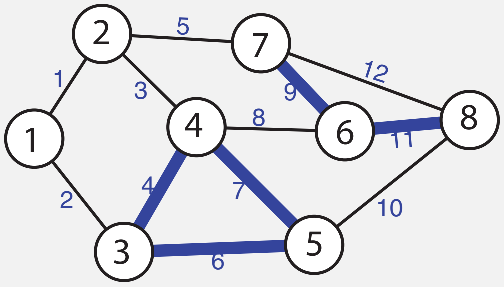

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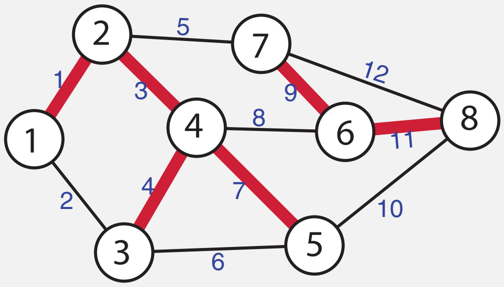
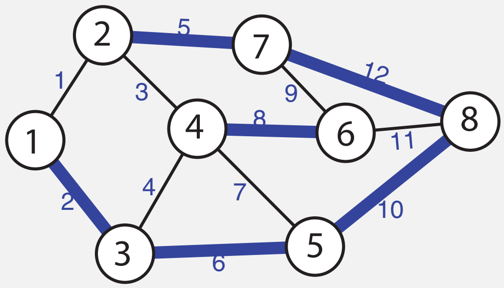
Independent but not spanning in M, and not closed in M.



Dependent in M* (contains a cycle, is a nonminimal cut)



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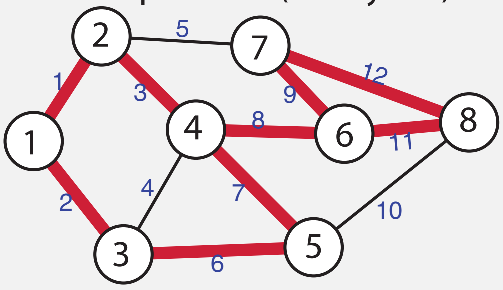
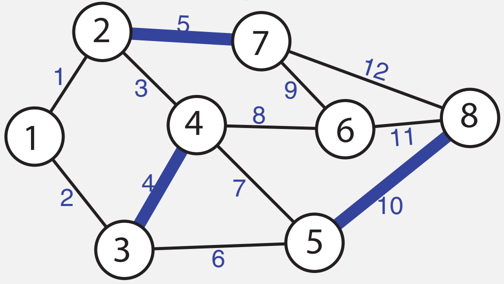
Spanning in M, but not a base, and not independent (has cycles)



Independent in M* (does not contain a cut)



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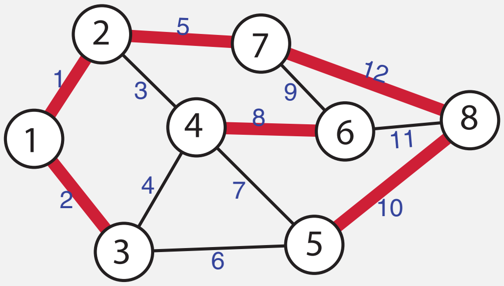
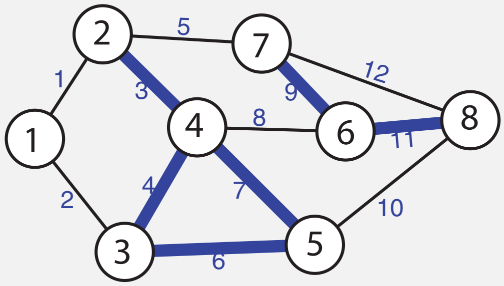


Dependent in M* (contains a cocycle, is a nonminimal cut)



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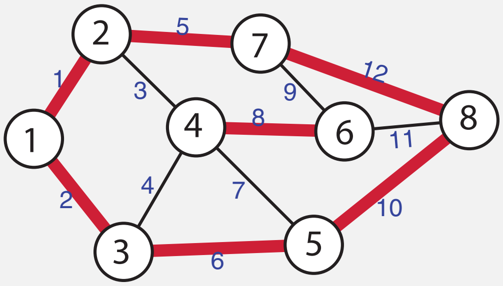
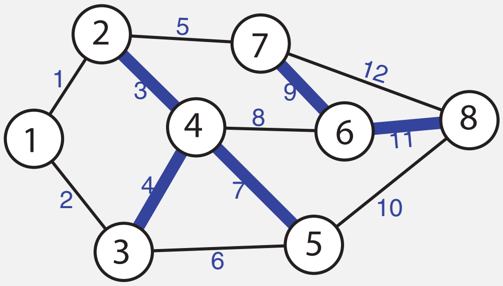
A hyperplane in M, dependent but not spanning in M



A cycle in M* (minimally dependent in M*, a cocycle, or a minimal cut)



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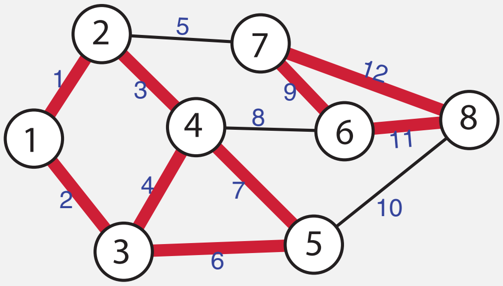

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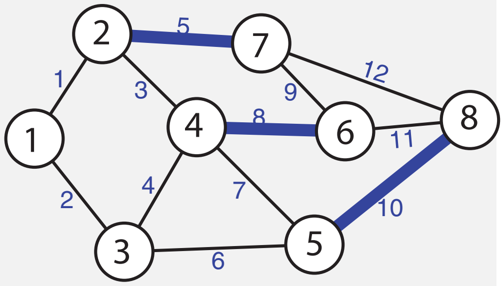

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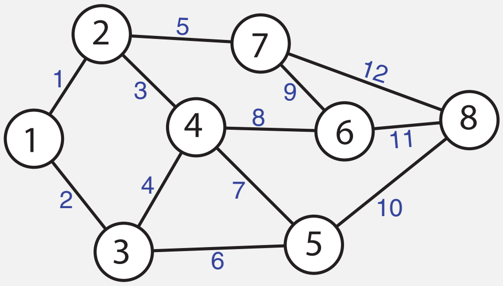
Cycle Matroid - independent sets have no cycles.



Cocycle matroid, independent sets contain no cuts.