Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture 8 —

http://www.ee.washington.edu/people/faculty/bilmes/classes/ee563_spring_2018/

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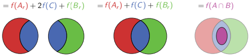
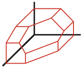
April 18th, 2018

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$





Cumulative Outstanding Reading

- Read chapter 1 from Fujishige's book.
- Read chapter 2 from Fujishige's book.

Announcements, Assignments, and Reminders

- If you have any questions about anything, please ask then via our discussion board (https://canvas.uw.edu/courses/1216339/discussion_topics).

Class Road Map - EE563

- L1(3/26): Motivation, Applications, & Basic Definitions,
- L2(3/28): Machine Learning Apps (diversity, complexity, parameter, learning target, surrogate).
- L3(4/2): Info theory exs, more apps, definitions, graph/combinatorial examples
- L4(4/4): Graph and Combinatorial Examples, Matrix Rank, Examples and Properties, visualizations
- L5(4/9): More Examples/Properties/ Other Submodular Defs., Independence,
- L6(4/11): Matroids, Matroid Examples, Matroid Rank, Partition/Laminar Matroids
- L7(4/16): Laminar Matroids, System of Distinct Reps, Transversals, Transversal Matroid, Matroid Representation, Dual Matroids
- L8(4/18): Dual Matroids, Other Matroid Properties, Combinatorial Geometries, Matroids and Greedy.
- L9(4/23):
- L10(4/25):
- L11(4/30):
- L12(5/2):
- L13(5/7):
- L14(5/9):
- L15(5/14):
- L16(5/16):
- L17(5/21):
- L18(5/23):
- L–(5/28): Memorial Day (holiday)
- L19(5/30):
- L21(6/4): Final Presentations maximization.

Last day of instruction, June 1st. Finals Week: June 2-8, 2018.

System of Distinct Representatives

- Let $(V, \mathcal{V})$ be a set system (i.e., $\mathcal{V} = (V_i : i \in I)$ where $V_i \subseteq V$ for all i), and I is an index set. Hence, $|I| = |\mathcal{V}|$.
- A family $(v_i : i \in I)$ with $v_i \in V$ is said to be a system of distinct representatives of $\mathcal{V}$ if $\exists$ a bijection $\pi : I \leftrightarrow I$ such that $v_i \in V_{\pi(i)}$ and $v_i \neq v_j$ for all $i \neq j$.
- In a system of distinct representatives, there is a requirement for the representatives to be distinct. We can re-state (and rename) this as a:

Definition 8.2.1 (transversal)

Given a set system $(V, \mathcal{V})$ and index set I for $\mathcal{V}$ as defined above, a set $T \subseteq V$ is a transversal of $\mathcal{V}$ if there is a bijection $\pi : T \leftrightarrow I$ such that

$$x \in V_{\pi(x)} \text{ for all } x \in T \tag{8.2}$$

- Note that due to $\pi : T \leftrightarrow I$ being a bijection, all of I and T are "covered" (so this makes things distinct automatically).

When do transversals exist?

- As we saw, a transversal might not always exist. How to tell?
- Given a set system $(V, \mathcal{V})$ with $\mathcal{V} = (V_i : i \in I)$, and $V_i \subseteq V$ for all i. Then, for any $J \subseteq I$, let

$$V(J) = \cup_{j \in J} V_j \tag{8.2}$$

so $|V(J)| : 2^I \to \mathbb{Z}_+$ is the set cover func. (we know is submodular).
- We have

Theorem 8.2.1 (Hall's theorem)

Given a set system $(V, \mathcal{V})$, the family of subsets $\mathcal{V} = (V_i : i \in I)$ has a transversal $(v_i : i \in I)$ iff for all $J \subseteq I$

$$|V(J)| \geq |J| \tag{8.3}$$

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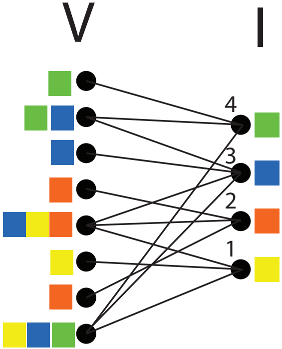
- Hall's theorem $(\forall J \subseteq I, |V(J)| \geq |J|)$ as a bipartite graph.



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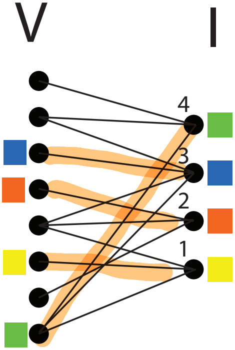

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- Moreover, we have

Theorem 8.2.2 (Rado's theorem (1942))

If $M = (V, r)$ is a matroid on V with rank function r, then the family of subsets $(V_i : i \in I)$ of V has a transversal $(v_i : i \in I)$ that is independent in M iff for all $J \subseteq I$

$$r(V(J)) \geq |J| \tag{8.4}$$

- Note, a transversal T independent in M means that $r(T) = |T|$.

Application's of Hall's theorem

- Consider a set of jobs I and a set of applicants V to the jobs. If an applicant $v \in V$ is qualified for job $i \in I$, we add edge (v, i) to the bipartite graph $G = (V, I, E)$.

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- Note if $|V| = |I|$, then Hall's theorem is the Marriage Theorem (Frobenious 1917), where an edge (v, i) in the graph indicate compatibility between two individuals $v \in V$ and $i \in I$ coming from two separate groups V and I.

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- If $\forall J \subseteq I, |V(J)| \geq |J|$, then all individuals in each group can be matched with a compatible mate.

More general conditions for existence of transversals

Theorem 8.2.1 (Polymatroid transversal theorem)

If $\mathcal{V} = (V_i : i \in I)$ *is a finite family of non-empty subsets of* V*, and* $f : 2^V \to \mathbb{Z}_+$ *is a non-negative, integral, monotone non-decreasing, and submodular function, then* $\mathcal{V}$ *has a system of representatives* $(v_i : i \in I)$ *such that*

$$f(\cup_{i \in J}\{v_i\}) \geq |J| \text{ for all } J \subseteq I \tag{8.2}$$

if and only if

$$f(V(J)) \geq |J| \text{ for all } J \subseteq I \tag{8.3}$$

- Given Theorem ??, we immediately get Theorem 8.2.1 by taking $f(S) = |S|$ for $S \subseteq V$.
- We get Theorem ?? by taking $f(S) = r(S)$ for $S \subseteq V$, the rank function of the matroid.

Review from Lecture 6

The next frame comes from lecture 6.

Matroids, other definitions using matroid rank $r : 2^V \to \mathbb{Z}_+$

Definition 8.3.3 (closed/flat/subspace)

A subset $A \subseteq E$ is closed (equivalently, a flat or a subspace) of matroid M if for all $x \in E \setminus A$, $r(A \cup \{x\}) = r(A) + 1$. (max'l set of a given rank).

Definition: A hyperplane is a flat of rank $r(M) - 1$.

Definition 8.3.4 (closure)

$r(A+b_1) + r(A+b_2) \geq r(A+b_1+b_2) + r(A)$

Given $A \subseteq E$, the closure (or span) of A, is defined by $\text{span}(A) = \{b \in E : r(A \cup \{b\}) = r(A)\}$.

$\Rightarrow r(A+b_1+b_2) = r(A)$

b_1, b_2 $r(A+b_1) = r(A)$, $r(A+b_2) = r(A)$

$r(A+b_1+b_2)$

$\Rightarrow r(\text{span}(A)) = r(A)$.

Therefore, a closed set A has $\text{span}(A) = A$.

Definition 8.3.5 (circuit)

A subset $A \subseteq E$ is circuit or a cycle if it is an inclusionwise-minimal dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).

$r(A) = |A| - 1$

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- A base of a matroid is a minimal spanning set (and it is independent) but supersets of a base are also spanning.
- V is always trivially spanning.
- Consider the terminology: "spanning tree in a graph", comes from spanning in a matroid sense.

Dual of a Matroid

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- We define the set of sets $\mathcal{I}^*$ for M^* as follows:

$$\mathcal{I}^* = \{A \subseteq V : V \setminus A \text{ is a spanning set of } M\} \tag{8.1}$$

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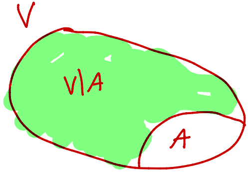
i.e., $\mathcal{I}^*$ are complements of spanning sets of M.



if V\A is spanning in M
then A is independent in M*

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- In other words, a set $A \subseteq V$ is independent in the dual M^* (i.e., $A \in \mathcal{I}^*$) if A's complement is spanning in M (residual $V \setminus A$ must contain a base in M).

$\exists$ base B^* of M^* s.t. $B^* \subseteq V \setminus A$ iff $A \in \mathcal{I}^*$

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- Dual of the dual: Note, we have that $(M^*)^* = M$.

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Theorem 8.3.3 (Dual matroid bases)

Let $M = (V, \mathcal{I})$ be a matroid and $\mathcal{B}(M)$ be the set of bases of M. Then define

$$\mathcal{B}^*(M) = \{V \setminus B : B \in \mathcal{B}(M)\}. \tag{8.4}$$

Then $\mathcal{B}^*(M)$ is the set of basis of M^* (that is, $\mathcal{B}^*(M) = \mathcal{B}(M^*)$).

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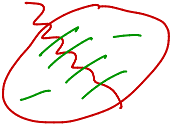

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- A minimal cut in G is a cut $X \subseteq E(G)$ such that $X \setminus \{x\}$ is not a cut for any $x \in X$.

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- All dependent sets in a cocycle matroid are cuts (i.e., a dependent set is a minimal cut or contains one).

Example: cocycle matroid (sometimes "cut matroid")

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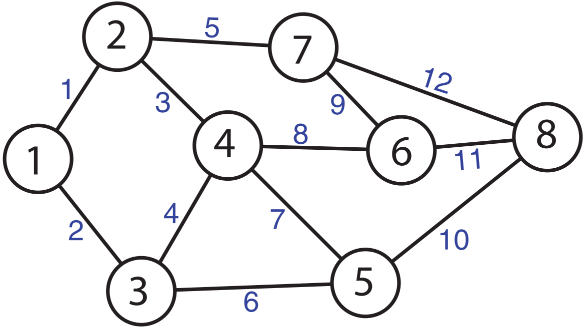
A graph G



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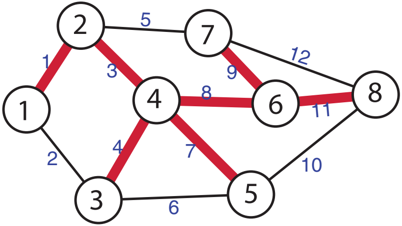
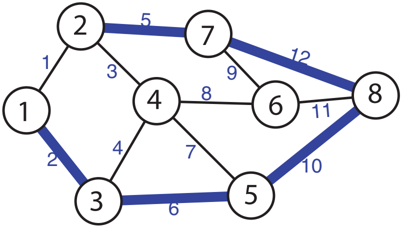


Maximally independent in M* (thus a base, minimally spanning, in M*)



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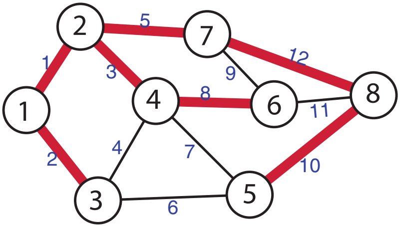

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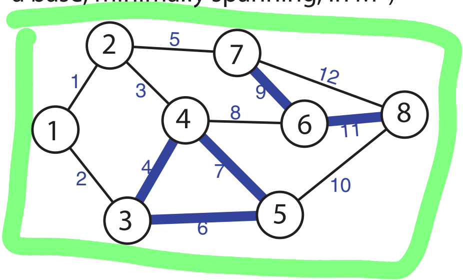

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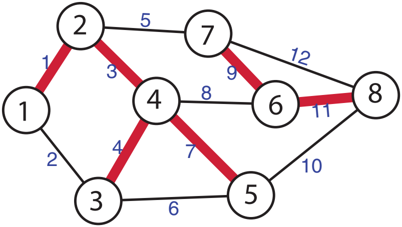
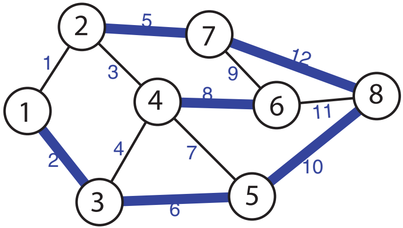
Independent but not spanning in M, and not closed in M.



Dependent in M* (contains a cocycle, is a nonminimal cut)



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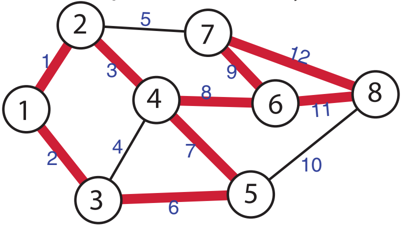
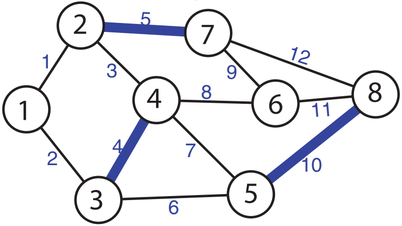
Spanning in M, but not a base, and not independent (has cycles)



Independent in M* (does not contain a cut)



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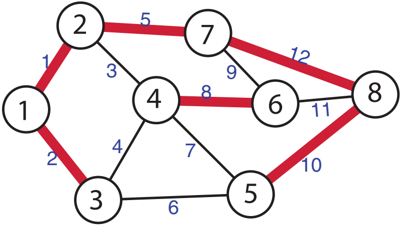

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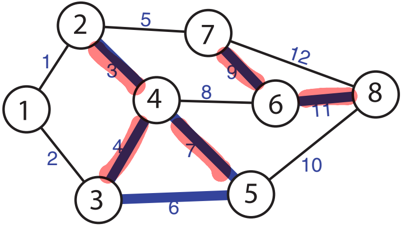

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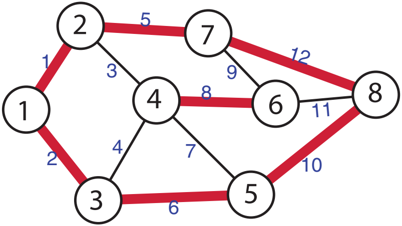
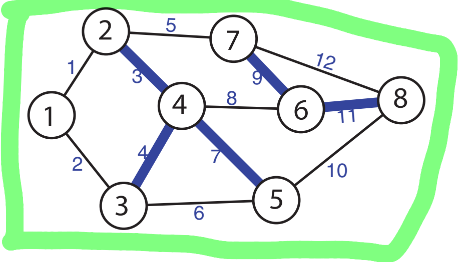
A hyperplane in M, dependent but not spanning in M



A cycle in M* (minimally dependent in M*, a cocycle, or a minimal cut)



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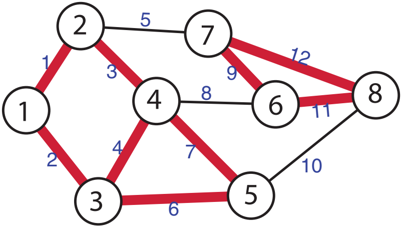

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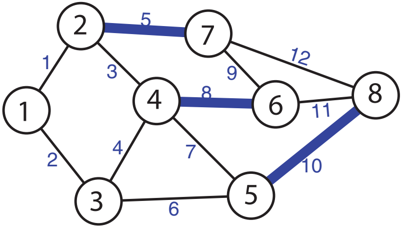

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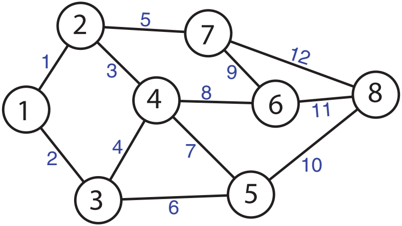
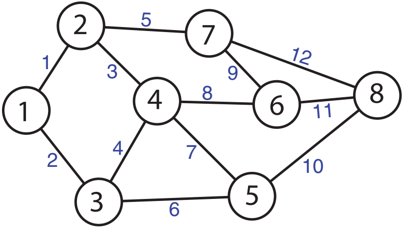
Cycle Matroid - independent sets have no cycles.



Cocycle matroid, independent sets contain no cuts.



The dual of a matroid is (indeed) a matroid

Theorem 8.3.5

Given matroid $M = (V, \mathcal{I})$, let $M^* = (V, \mathcal{I}^*)$ be as previously defined. Then M^* is a matroid.

Proof.

- Since $V \setminus \emptyset$ is spanning in primal, clearly $\emptyset \in \mathcal{I}^*$, so (I1') holds.

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- Also, if $I \subseteq J \in \mathcal{I}^*$, then clearly also $I \in \mathcal{I}^*$ since if $V \setminus J$ is spanning in M, so must $V \setminus I$. Therefore, (I2') holds. $(V \setminus J) \subseteq (V \setminus I)$
- Next, given $I, J \in \mathcal{I}^*$ with $|I| < |J|$, it must be the case that $\bar{I} = V \setminus I$ and $\bar{J} = V \setminus J$ are both spanning in M with $|\bar{I}| > |\bar{J}|$.

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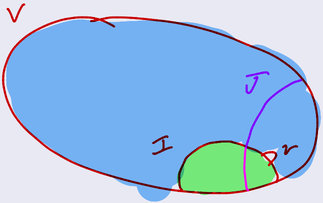

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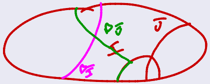
- Since $B_{\bar{J}} \setminus I \subseteq V \setminus I$, and $B_{\bar{J}} \setminus I$ is independent in M, we can choose the base $B_{\bar{I}}$ of M s.t. $B_{\bar{J}} \setminus I \subseteq B_{\bar{I}} \subseteq V \setminus I$.



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- Since $B_{\bar{J}}$ and J are disjoint, we have both: 1) $B_{\bar{J}} \setminus I$ and $J \setminus I$ are disjoint; and 2) $B_{\bar{J}} \cap I \subseteq I \setminus J$. Also note, $B_{\bar{I}}$ and I are disjoint.

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$|J| > |I|$

$$|B_{\bar{J}}| = |B_{\bar{J}} \cap I| + |B_{\bar{J}} \setminus I| \tag{8.5}$$
$$\leq |I \setminus J| + |B_{\bar{J}} \setminus I| \tag{8.6}$$
$$< |J \setminus I| + |B_{\bar{J}} \setminus I| \leq |B_{\bar{I}}| \tag{8.7}$$

which is a contradiction. *The last inequality on the right follows since $J \setminus I \subseteq B_{\bar{I}}$ (by assumption) and $B_{\bar{J}} \setminus I \subseteq B_{\bar{I}}$ implies that $(J \setminus I) \cup (B_{\bar{J}} \setminus I) \subseteq B_{\bar{I}}$, but since J and $B_{\bar{J}}$ are disjoint, we have that $|J \setminus I| + |B_{\bar{J}} \setminus I| \leq |B_{\bar{I}}|$.*

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  which is a contradiction.
- Therefore, $J \setminus I \not\subseteq B_{\bar{I}}$, and there is a $v \in J \setminus I$ s.t. $v \notin B_{\bar{I}}$.
- So $B_{\bar{I}}$ is disjoint with $I \cup \{v\}$, means $B_{\bar{I}} \subseteq V \setminus (I \cup \{v\})$, or $V \setminus (I \cup \{v\})$ is spanning in M, and therefore $I \cup \{v\} \in \mathcal{I}^*$.

□

Matroid Duals and Representability

Theorem 8.3.6

Let M be an $\mathbb{F}$-representable matroid (i.e., one that can be represented by a finite sized matrix over field $\mathbb{F}$). Then M^* is also $\mathbb{F}$-representable.

Hence, for matroids as general as matric matroids, duality does not extend the space of matroids that can be used.

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Hence, for matroids as general as matric matroids, duality does not extend the space of matroids that can be used.

Theorem 8.3.7

Let M be a graphic matroid (i.e., one that can be represented by a graph $G = (V, E)$). Then M^* is not necessarily also graphic.

Hence, for graphic matroids, duality can increase the space and power of matroids, and since they are based on a graph, they are relatively easy to use: 1) all cuts are dependent sets; 2) minimal cuts are cycles; 3) bases of a cut are any one edge removed from minimal cuts; 4) independent sets are edges that are not cuts (minimal or otherwise); 5) bases of matroid are maximal non-cuts (non-cut containing edge sets).

Dual Matroid Rank

Theorem 8.3.8

The rank function r_{M^*} of the dual matroid M^* may be specified in terms of the rank r_M in matroid M as follows. For $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) \tag{8.8}$$

- Note, we again immediately see that this is submodular by the properties of submodular functions we saw in lectures 1 and 2. *I.e., $|X|$ is modular, complement $f(V \setminus X)$ is submodular if f is submodular, $r_M(V)$ is a constant, and summing submodular functions and a constant preserves submodularity.*

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- Non-negativity integral follows since $|X| + r_M(V \setminus X) \geq r_M(X) + r_M(V \setminus X) \geq r_M(V)$. The right inequality follows since r_M is submodular.

$+ r_M(\phi)$ $= 0$

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- Therefore, r_{M^*} is the rank function of a matroid. That it is the dual matroid rank function is shown in the next proof.

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Proof.

A set X is independent in (V, r_{M^*}) if and only if

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) = |X| \tag{8.9}$$

...

Dual Matroid Rank

Theorem 8.3.8

The rank function r_{M^*} of the dual matroid M^* may be specified in terms of the rank r_M in matroid M as follows. For $X \subseteq V$:

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Proof.

A set X is independent in (V, r_{M^*}) if and only if

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But a subset X is independent in M^* only if $V \setminus X$ is spanning in M (by the definition of the dual matroid). □

Matroid restriction/deletion

- Let $M = (V, \mathcal{I})$ be a matroid and let $Y \subseteq V$, then

$$\mathcal{I}_Y = \{Z : Z \subseteq Y, Z \in \mathcal{I}\} \tag{8.11}$$

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$(Z \cap X = \phi)$
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- Hence, $M|Y = M \setminus (V \setminus Y)$, and $M|(V \setminus X) = M \setminus X$.
- The rank function is of the same form. I.e., $r_Y : 2^Y \to \mathbb{Z}_+$, where $r_Y(Z) = r(Z)$ for $Z \subseteq Y$, $Y = V \setminus X$.

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$I \cap B_Z = \phi$

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- A minor of a matroid is any matroid obtained via a series of deletions and contractions of some matroid.
- In fact, it is the case $M/Z = (M^* \setminus Z)^*$ (Exercise: show why).

Matroid Intersection

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This is an instance of the convolution of two submodular functions, f_1 and f_2 that, evaluated at $Y \subseteq V$, is written as:

$$(f_1 * f_2)(Y) = \min_{X \subseteq Y} \Big(f_1(X) + f_2(Y \setminus X) \Big) \tag{8.16}$$

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- So Hall's theorem can be expressed as convolution. Exercise: define $g(A) = [\Gamma(\cdot) * |\cdot|](A)$, prove that g is submodular.
- Note, in general, convolution of two submodular functions does not preserve submodularity (but in certain special cases it does).

Matroid Union

Definition 8.4.2

Let $M_1 = (V_1, \mathcal{I}_1)$, $M_2 = (V_2, \mathcal{I}_2)$, $\dots$, $M_k = (V_k, \mathcal{I}_k)$ be matroids. We define the union of matroids as $M_1 \vee M_2 \vee \dots \vee M_k = (V_1 \uplus V_2 \uplus \dots \uplus V_k, \mathcal{I}_1 \vee \mathcal{I}_2 \vee \dots \vee \mathcal{I}_k)$, where

$$I_1 \vee \mathcal{I}_2 \vee \dots \vee \mathcal{I}_k = \{I_1 \uplus I_2 \uplus \dots \uplus I_k | I_1 \in \mathcal{I}_1, \dots, I_k \in \mathcal{I}_k\} \quad (8.17)$$

Note $A \uplus B$ designates the disjoint union of A and B.

$V_1 = \{a, b, c\}$ $\quad V_2 = \{a, b, c\}$

$V_1 \uplus V_2 = \{ (v, i) : v \in \{a, b, c\}, i \in \{1, 2\} \}$

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Theorem 8.4.3

Let $M_1 = (V_1, \mathcal{I}_1)$, $M_2 = (V_2, \mathcal{I}_2)$, $\ldots$, $M_k = (V_k, \mathcal{I}_k)$ be matroids, with rank functions $r_1, \ldots, r_k$. Then the union of these matroids is still a matroid, having rank function

$$r(Y) = \min_{X \subseteq Y} \Big(|Y \setminus X| + r_1(X \cap V_1) + \cdots + r_k(X \cap V_k) \Big) \tag{8.18}$$

for any $Y \subseteq V_1 \uplus \ldots V_2 \uplus \cdots \uplus V_k$.

Exercise: Matroid Union, and Matroid duality

Exercise: Fully characterize $M \vee M^*$.

Matroids of three or fewer elements are graphic

- All matroids up to and including three elements (edges) are graphic.

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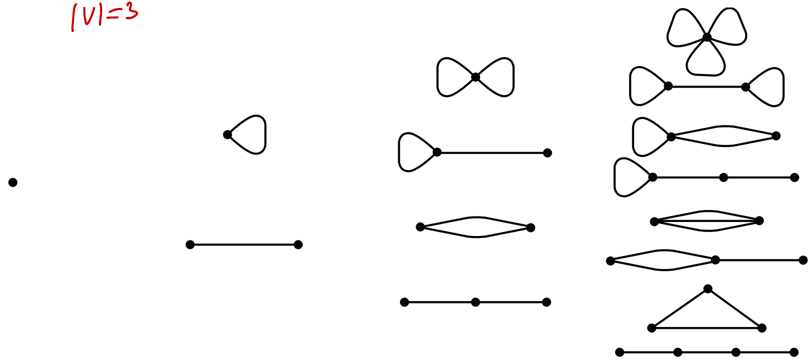
$|V|=3$



(a) The only matroid with zero elements.

(b) The two one-element matroids.

(c) The four two-element matroids.

(d) The eight three-element matroids.

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$M = (V, \mathcal{I})$

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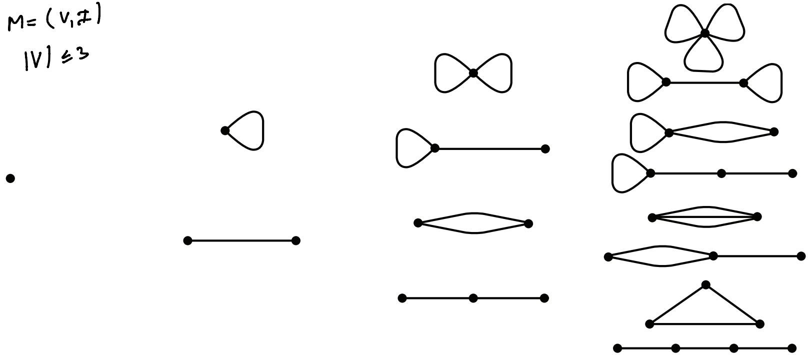

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- This is a nice way to visualize matroids with very low ground set sizes. What about matroids that are low rank but with many elements?

$r(M)=3$

$|V|$ big.

Affine Matroids

- Given an $n \times m$ matrix with entries over some field $\mathbb{F}$, we say that a subset $S \subseteq \{1, \dots, m\}$ of indices (with corresponding column vectors $\{v_i : i \in S\}$, with $|S| = k \leq m$) is affinely dependent if $m \geq 1$ and there exists elements $\{a_1, \dots, a_k\} \in \mathbb{F}$, not all zero with $\sum_{i=1}^k a_i = 0$, such that $\sum_{i=1}^k a_i v_i = 0$.

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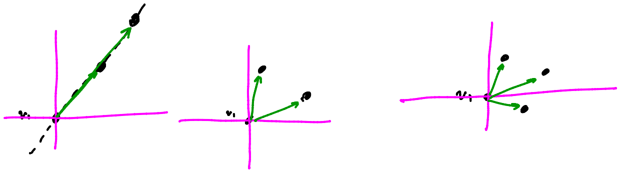
- Example: in 2D, three collinear points are affinely dependent, three non-collear points are affinely independent, and ≥ 4 collinear or non-collinear points are affinely dependent.



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Exercise: prove this.

Euclidean Representation of Low-rank Matroids

- Consider the affine matroid with $n \times m = 2 \times 6$ matrix on the field $\mathbb{F} = \mathbb{R}$, and let the elements be $\{(0,0), (1,0), (2,0), (0,1), (0,2), (1,1)\}$.

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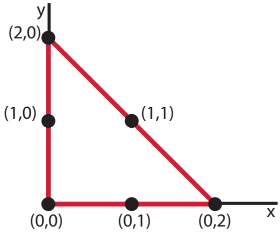
- We can plot the points in $\mathbb{R}^2$ as on the right:



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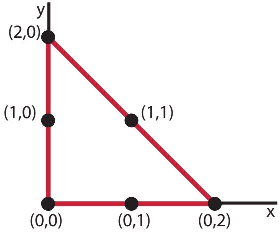

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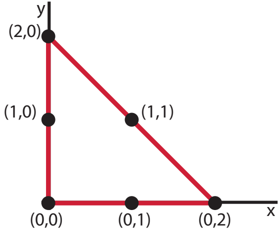

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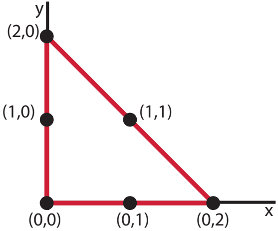

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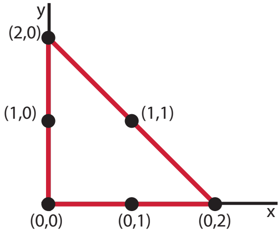

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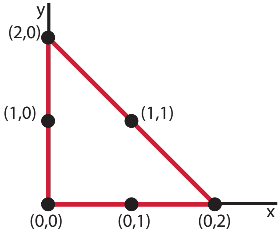
- Dependent sets consist of all subsets with ≥ 4 elements (rank 3), or 3 collinear elements (rank 2). Any two points have rank 2.



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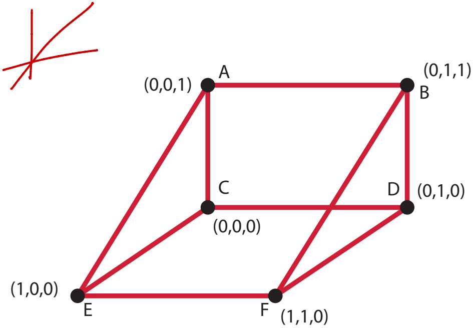

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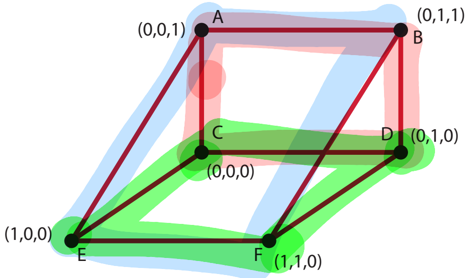

- All sets of 5 points are dependent. The only other sets of dependent points are coplanar ones of size 4. Namely:
  $\{(0,0,0),(0,1,0),(1,1,0),(1,0,0)\}$,
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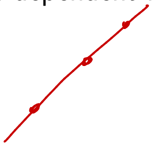

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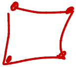

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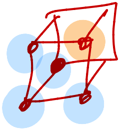

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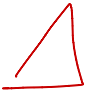

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Theorem 8.5.2

Any matroid of rank $m \leq 4$ can be represented by an affine matroid in $\mathbb{R}^{m-1}$. regardless of how big |V| is.

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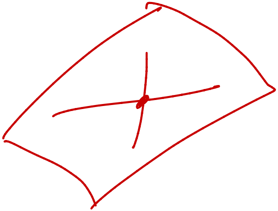

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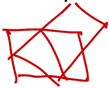

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- (see Oxley 2011 for more details).

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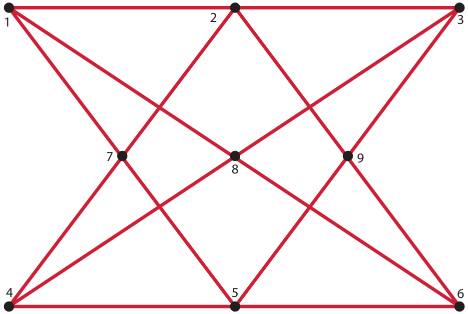

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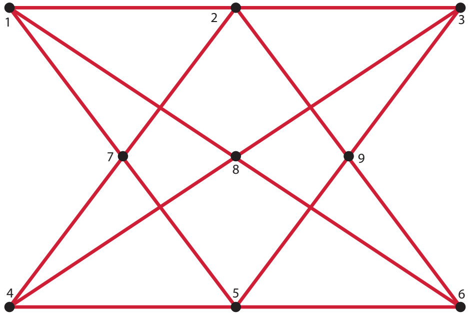

- Called the non-Pappus matroid. Has rank three, but any matric matroid with the above dependencies would require that $\{7, 8, 9\}$ is dependent, hence requiring an additional line in the above.

Euclidean Representation of Low-rank Matroids: A test

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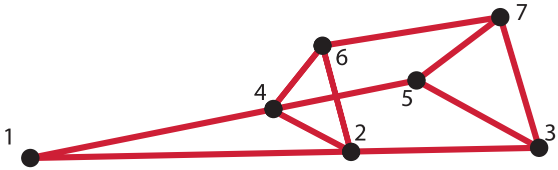

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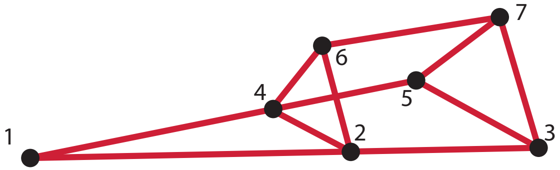

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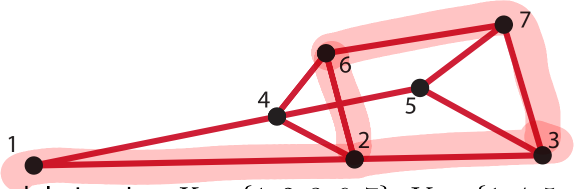

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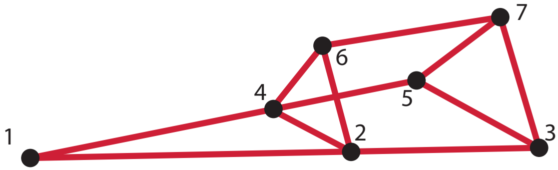

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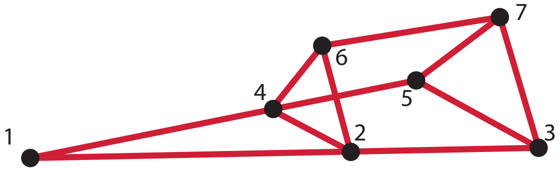

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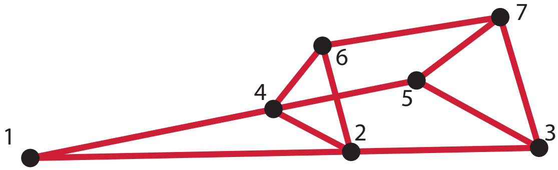

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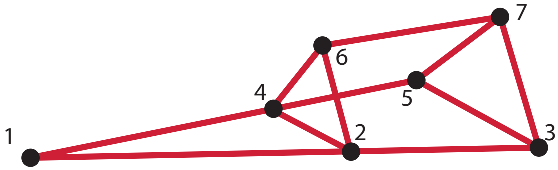

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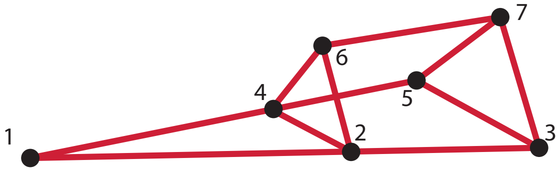

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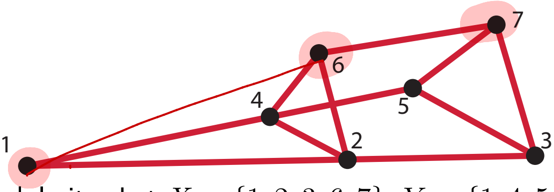

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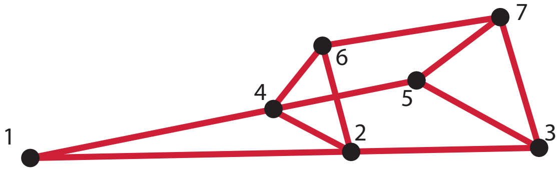

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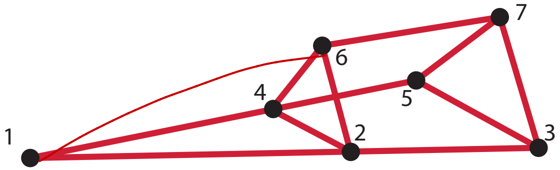

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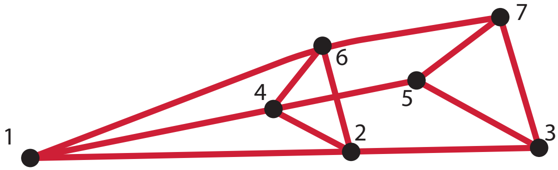
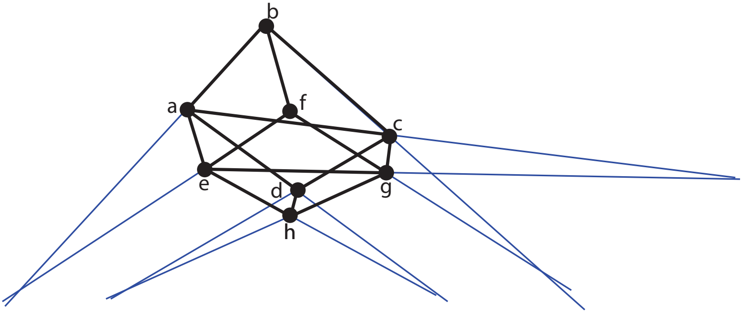


- If we extend the line from 6-7 to 1, then is it a matroid?
- Hence, not all 2D or 3D graphs of points and lines are matroids.

Matroid?

- Consider the following geometry on $|V| = 8$ points with $V = \{a, b, c, d, e, f, g, h\}$.



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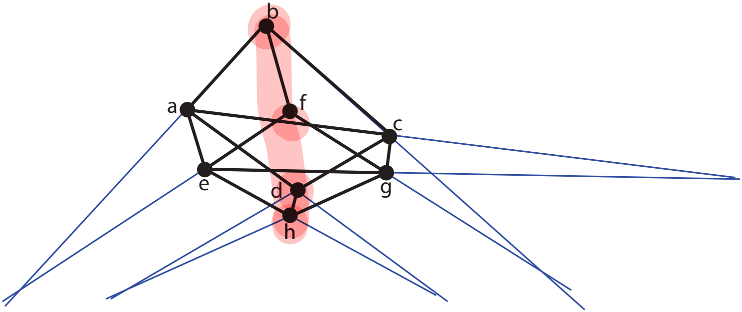

- Note, we are given that the points $\{b, d, h, f\}$ are not coplanar. However, the following sets of points are coplanar: $\{a, b, e, f\}$, $\{d, c, g, h\}$, $\{a, d, h, e\}$, $\{b, c, g, f\}$, $\{b, c, d, a\}$, $\{f, g, h, e\}$, and $\{a, c, g, e\}$.

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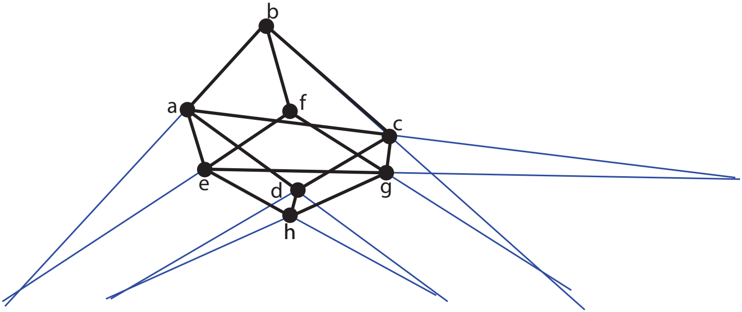

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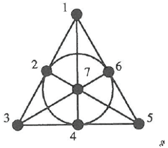
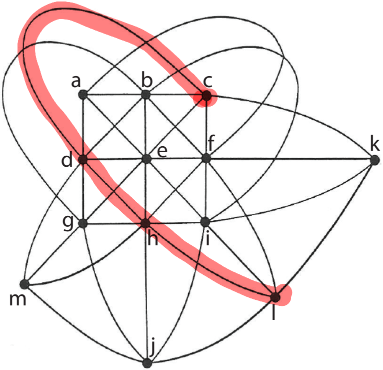
- Exercise: Is this a matroid? Exercise: If so, is it representable?

Projective Geometries: Other Examples

- Other examples can be more complex, consider the following two matroids (from Oxley, 2011):





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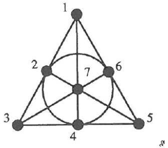

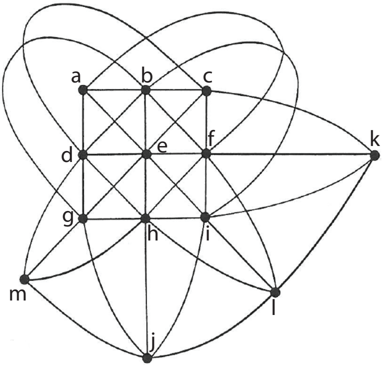

- Right: a matroid (and a 2D depiction of a geometry) over the field $\text{GF}(3) = \{0, 1, 2\}$ mod 3 and is "coordinatizable" in $\text{GF}(3)^3$.

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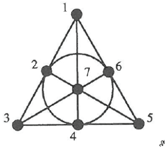

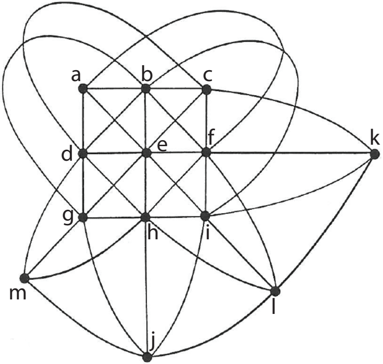

- Right: a matroid (and a 2D depiction of a geometry) over the field $\mathrm{GF}(3) = \{0, 1, 2\}$ mod 3 and is "coordinatizable" in $\mathrm{GF}(3)^3$.
- Hence, lines (in 2D) which are rank 2 sets may be curved; planes (in 3D) can be twisted.

Matroids, Representation and Equivalence: Summary

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- Not all matroids are linear (i.e., matric) matroids.
- Matroids can be seen as related to projective geometries (and are sometimes called combinatorial geometries).
- Exists much research on different subclasses of matroids, and if/when they are contained in (or isomorphic to) each other.

Matroid Further Reading

- "Matroids: A Geometric Introduction", Gordon and McNulty, 2012.
- "The Coming of the Matroids", William Cunningham, 2012 (a nice history)
- Welsh, "Matroid Theory", 1975.
- Oxley, "Matroid Theory", 1992 (and 2011) (perhaps best "single source" on matroids right now).
- Crapo & Rota, "On the Foundations of Combinatorial Theory: Combinatorial Geometries", 1970 (while this is old, it is very readable).
- Lawler, "Combinatorial Optimization: Networks and Matroids", 1976.
- Schrijver, "Combinatorial Optimization", 2003

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- Greedy is good since it can be made to run very fast $O(n \log n)$.
- Often, however, greedy is heuristic (it might work well in practice, but worst-case performance can be unboundedly poor).
- We will next see that the greedy algorithm working optimally is a defining property of a matroid, and is also a defining property of a polymatroid function.

Matroid and the greedy algorithm

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Algorithm 1: The Matroid Greedy Algorithm

Set $X \leftarrow \emptyset$;
while $\exists v \in E \setminus X$ *s.t.* $X \cup \{v\} \in \mathcal{I}$ **do**
    $v \in \operatorname{argmax} \{w(v) : v \in E \setminus X,\ X \cup \{v\} \in \mathcal{I}\}$;
    $X \leftarrow X \cup \{v\}$;

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```
Set X ← ∅ ;
while ∃v ∈ E \ X s.t. X ∪ {v} ∈ I do
    v ∈ argmax {w(v) : v ∈ E \ X, X ∪ {v} ∈ I} ;
    X ← X ∪ {v} ;
```

- Same as sorting items by decreasing weight w, and then choosing items in that order that retain independence.

Matroid and the greedy algorithm

empty set containing & down closed.

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Theorem 8.6.1

Let $(E, \mathcal{I})$ be an independence system. Then the pair $(E, \mathcal{I})$ is a matroid if and only if for each weight function $w \in \mathcal{R}_+^E$, Algorithm 1 above leads to a set $I \in \mathcal{I}$ of maximum weight $w(I)$.

Review from Lecture 6

- The next slide is from Lecture 6.

Matroids by bases

In general, besides independent sets and rank functions, there are other equivalent ways to characterize matroids.

Theorem 8.6.3 (Matroid (by bases))

Let E be a set and $\mathcal{B}$ be a nonempty collection of subsets of E. Then the following are equivalent.

1. *$\mathcal{B}$ is the collection of bases of a matroid;*
2. *if $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B' - x + y \in \mathcal{B}$ for some $y \in B \setminus B'$.*
3. *If $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B - y + x \in \mathcal{B}$ for some $y \in B \setminus B'$.*

Properties 2 and 3 are called "exchange properties."

Proof here is omitted but think about this for a moment in terms of linear spaces and matrices, and (alternatively) spanning trees.

Matroid and the greedy algorithm

proof of Theorem 8.6.1.

- Assume $(E, \mathcal{I})$ is a matroid and $w : E \to \mathcal{R}_+$ is given.

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- Assume $(E, \mathcal{I})$ is a matroid and $w : E \to \mathcal{R}_+$ is given.
- Let $A = (a_1, a_2, \dots, a_r)$ be the solution returned by greedy, where $r = r(M)$ the rank of the matroid, and we order the elements as they were chosen (so $w(a_1) \geq w(a_2) \geq \cdots \geq w(a_r)$).

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- A is a base of M, and let $B = (b_1, \dots, b_r)$ be <u>any</u> another base of M with elements also ordered decreasing by weight, so $w(b_1) \geq w(b_2) \geq \dots \geq w(b_r)$.

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- We next show that not only is $w(A) \geq w(B)$ but that $w(a_i) \geq w(b_i)$ for all i.

...

Matroid and the greedy algorithm

proof of Theorem 8.6.1.

- Assume otherwise, and let k be the first (smallest) integer such that $w(a_k) < w(b_k)$. Hence $w(a_j) \geq w(b_j)$ for $j < k$.

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Matroid and the greedy algorithm

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- Assume otherwise, and let k be the first (smallest) integer such that $w(a_k) < w(b_k)$. Hence $w(a_j) \geq w(b_j)$ for $j < k$.
- Define independent sets $A_{k-1} = \{a_1, \ldots, a_{k-1}\}$ and $B_k = \{b_1, \ldots, b_k\}$.

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- Since $|A_{k-1}| < |B_k|$, there exists a $b_i \in B_k \setminus A_{k-1}$ where $A_{k-1} \cup \{b_i\} \in \mathcal{I}$ for some $1 \leq i \leq k$.

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- But $w(b_i) \geq w(b_k) > w(a_k)$, and so the greedy algorithm would have chosen b_i rather than a_k, contradicting what greedy does.

□

Matroid and the greedy algorithm

converse proof of Theorem 8.6.1.

- Given an independence system $(E, \mathcal{I})$, suppose the greedy algorithm leads to an independent set of max weight for every non-negative weight function. We'll show $(E, \mathcal{I})$ is a matroid.

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- Let $I, J \in \mathcal{I}$ with $|I| < |J|$. Suppose to the contrary, that $I \cup \{z\} \notin \mathcal{I}$ for all $z \in J \setminus I$.

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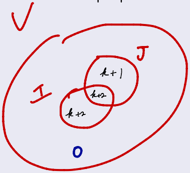
- Let $I, J \in \mathcal{I}$ with $|I| < |J|$. Suppose to the contrary, that $I \cup \{z\} \notin \mathcal{I}$ for all $z \in J \setminus I$.
- Define the following modular weight function w on E, and define $k = |I|$.



$$w(v) = \begin{cases} k+2 & \text{if } v \in I, \\ k+1 & \text{if } v \in J \setminus I, \\ 0 & \text{if } v \in E \setminus (I \cup J) \end{cases} \tag{8.19}$$

...

Matroid and the greedy algorithm

converse proof of Theorem 8.6.1.

- Now greedy will, after k iterations, recover I, but it cannot choose any element in $J \setminus I$ by assumption. Thus, greedy chooses a set of weight $k(k+2) = w(I)$.

$|I| \cdot (k+2) = w(I)$

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- Now greedy will, after k iterations, recover I, but it cannot choose any element in $J \setminus I$ by assumption. Thus, greedy chooses a set of weight $k(k+2) = w(I)$.
- On the other hand, J has weight $|J| > |I| = k$

$$w(J) \geq |J|(k+1) \geq (k+1)(k+1) > k(k+2) = w(I) \qquad (8.20)$$

so J has strictly larger weight but is still independent, contradicting greedy's optimality.

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so J has strictly larger weight but is still independent, contradicting greedy's optimality.
- Therefore, there must be a $z \in J \setminus I$ such that $I \cup \{z\} \in \mathcal{I}$, and since I and J are arbitrary, $(E, \mathcal{I})$ must be a matroid.

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- Exercise: what if we keep going until a base even if we encounter negative values?

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- We don't need non-negativity, we can use any $w \in \mathbb{R}^E$ and keep going until we have a base.
- If we stop at a negative value, we'll once again get a maximum weight independent set.
- Exercise: what if we keep going until a base even if we encounter negative values?
- We can instead do as small as possible thus giving us a minimum weight independent set/base.

Summary of Important (for us) Matroid Definitions

Given an independence system, matroids are defined equivalently by any of the following:

- All maximally independent sets have the same size.
- A monotone non-decreasing submodular integral rank function with unit increments.
- The greedy algorithm achieves the maximum weight independent set for all weight functions.