EE 529 Homework #2 Solutions

1. (20%) The figure on the right shows the spontaneous emission spectrum of GaAs in thermal equilibrium at 300 K as an example.
   (a) Determine a general expression for the energy, in terms of bandgap energy $E_g$ and $k_B T$, at which the spontaneous emission spectrum $R_{sp}(E)$ reaches a peak value. Assume the semiconductor is in thermal equilibrium or under low injection/optical excitation.
   (b) Calculate its full width at half-maximum (FWHM).
   (c) Based on the result in (a), determine the peak emission wavelength if this is a GaAs LED operated at 300 K.

(a) $R_{sp}(E) = \frac{(2m^*_{s})^{3/2}}{2\pi^2\hbar^3\tau_{sp}}(E - E_{g})^{1/2}\exp\left(\frac{E_{F_c} - E_{F_v} - E_{g}}{k_B T}\right)\exp\left(-\frac{E - E_{g}}{k_B T}\right)$

This equation can be re-written as:

$$R_{sp}(E) = A(E - E_{g})^{1/2}\exp\left(-\frac{E - E_{g}}{k_B T}\right), \text{ where } A = \frac{(2m^*_{s})^{3/2}}{2\pi^2\hbar^3\tau_{sp}}\exp\left(\frac{E_{F_c} - E_{F_v} - E_{g}}{k_B T}\right).$$

Substituting $x = \frac{E - E_{g}}{k_B T}$, we get $f(x) \equiv R_{sp}(x) = A(k_B T)^{1/2}x^{1/2}e^{-x}$

Differentiating $f(x)$ with respect to $x$ and equating to zero to find the maxima, we get

$$x_{\text{max}} = \frac{1}{2}, \text{ or } E_{\text{max}} = E_{g} + \frac{k_B T}{2}.$$  

(b) Substituting the result from (a) into the spontaneous emission spectrum,

$$R_{sp}(E_{\text{max}}) = A\left(\frac{k_B T}{2}\right)^{1/2}e^{-1/2}$$

The FWHM is obtained from $R_{sp}(E_{i,2}) = \frac{1}{2}R_{sp}(E_{\text{max}})$, where $E_i$ and $E_2$ are the two solutions and FWHM $\equiv (E_2 - E_i)$. In terms of the corresponding $x$,

$$x^{1/2}e^{-x} - \frac{1}{2\sqrt{\pi}}e^{-1/2} = 0$$

Solving this equation numerically, we obtain

$x_1 = 0.051, \ x_2 = 1.846$

$\therefore E_2 - E_1 = (x_2 - x_1)k_B T = 1.8k_B T$

(c) $E_g = 1.424$ eV for GaAs at room temperature. The peak emission photon energy is

$$h\nu = E_g + \frac{1}{2}k_B T = 1.424 + 0.013 = 1.437 \text{ eV}$$

The peak emission wavelength
\[ \lambda_0 = \frac{1.24}{1.437} = 0.863 \text{ \(\mu\)m} \]

2. (20%) Consider direct band-to-band optical transitions in GaAs at \( \lambda = 800 \text{ nm} \) wavelength at 300 K. The effective mass for electrons and holes are \( m_e^* = 0.067m_0 \) and \( m_h^* = 0.52m_0 \).

(a) Find the energy levels \( E_2 \) and \( E_1 \), with respect to \( E_C \) and \( E_V \), for the optical transitions at this wavelength.

(b) Calculate the value of the density of states \( \rho(\nu) \) for these transitions.

(c) By taking \( \tau_{sp} = 500 \text{ ps} \), find the spectral spontaneous emission rate \( R_{sp}(\nu) \) for intrinsic GaAs at this optical wavelength.

(Note: You can compare your results with Example 13.2 to check if they are reasonable.)

(a) \[
\begin{align*}
\frac{m_e^*}{m_e} + \frac{m_h^*}{m_h} &= \frac{0.067 \times 0.52}{0.067 + 0.52} m_0 = 0.0594m_0 \\
\rho(\nu) &= \frac{4\pi(2m_e^*)^{3/2}}{\hbar^2} \left( \nu - E_g \right)^{1/2} \\
&= \frac{4\pi(2 \times 0.0594 \times 9.11 \times 10^{-31})^{3/2}}{(6.62 \times 10^{-34})^2} \left[ (1.55 - 1.424)1.6 \times 10^{-19} \right]^{1/2} = 1.45 \times 10^{11} \text{ m}^{-3}\text{Hz}^{-1}
\end{align*}
\]

Compared to that found in Example 13.2, the density of states increases at the higher photon energy as expected.

(c) \( R_{sp}(\nu) = \frac{1}{\tau_{sp}}f_c(E_2)[1 - f_v(E_1)]\rho(\nu) \)

For intrinsic GaAs at 300 K,

\[
\begin{align*}
f_c(E_2) &\approx e^{(E_p - E_2)/k_BT} \\
f_v(E_1) &\approx e^{(E_1 - E_F)/k_BT} \\
f_c(E_2)[1 - f_v(E_1)] &\approx e^{(E_1 - E_2)/k_BT} = e^{-\nu/k_BT}
\end{align*}
\]

\[
R_{sp}(\nu) = \frac{1}{500 \times 10^{-12}} e^{-1.55/0.0259} \times 1.45 \times 10^{11} = 2.96 \times 10^{-6} \text{ m}^{-3}
\]

This is more than one order of magnitude smaller than that found in Example 13.2 because of the negative exponential dependence of \( R_{sp}(\nu) \) on the photon energy. Also, from the spontaneous emission spectrum shown in Problem 1, this photon energy is further away from the peak of the spontaneous emission spectrum.
3. (20%) Franz-Keldysh effect in GaAs
   (a) Use what we learned about the Franz-Keldysh effect in class, plot a figure for absorption coefficient of GaAs under various electric fields like the one shown in Slide 10 of the supplementary slides. Start your plot at $E_g - h\omega = 0.001\text{ eV}$, $K = 5 \times 10^4 \text{ cm}^1(\text{eV})^{-1/2}$ in GaAs. (Note: Your results will look somewhat different from the figure in Slide 10, which is obtained using a different equation.)
   (b) Calculate the tunneling probability of an electron in GaAs under an applied field of $1 \times 10^5 \text{ V/cm}$ when a photon of 1.4 eV is absorbed at room temperature. Compare this with the band-to-band tunneling probability of an electron in the valence band without photon absorption with the same applied field.
   (c) We want to make an optical modulator utilizing the Franz-Keldysh effect in GaAs. Assume the applied electric field is $2 \times 10^5 \text{ V/cm}$, and we want to achieve 90% modulation depth, i.e., changing the output light intensity from full value to 10% of its full value. Calculate the required device length versus wavelength between 0.88 and 1 µm and plot your result in a figure.

   (a) The electric field-dependent absorption coefficient due to the Franz-Keldysh effect:
   $$\alpha = K(E')^{1/2}(8\beta)^{-1}\exp\left(-\frac{4}{3}\beta^{3/2}\right),$$
   where
   $$E' = \left(\frac{q^2E^2\hbar^2}{2m_r^*}\right)^{1/3}, \quad \beta = \frac{E_g - h\omega}{E'}$$
   The results are plotted below.

   (b) The tunneling probability of an electron with photon absorption is given by the exponential factor in the electric field-dependent absorption coefficient. This probability can therefore be expressed as
   $$\exp\left[-\frac{4\sqrt{2m_r^*}(E_g - h\omega)^{3/2}}{3e\hbar E}\right]$$
   where the reduced effective mass $m_r^*$ in GaAs is...
\[ m_r = \frac{m_e^* m_h^*}{m_e^* + m_h^*} = \frac{0.067 \times 0.52}{0.067 + 0.52} m_0 = 0.0594 m_0 \]

The value of the exponent is then (using MKS unit)
\[ 4/2 \times 0.056 \times 9.1 \times 10^{-31} \left( (1.424 - 1.4) \times 10^{-19} \right)^{3/2} \]
\[ = \frac{3.16 \times 10^{-19} \times 1.05 \times 10^{-34} \times 1 \times 10^7}{3.16 \times 10^{-19} \times 1.05 \times 10^{-34} \times 1 \times 10^7} = -0.62 \]

Therefore, tunneling probability = \( e^{-0.62} = 0.54 \). This is a very high probability for tunneling.

On the other hand, the band-to-band tunneling probability without the assistance of photon absorption is given by
\[ \exp \left[ -\frac{4\sqrt{2} \times 0.056 \times 9.1 \times 10^{-31} \left( 1.424 \times 1.6 \times 10^{-19} \right)^{3/2}}{3.16 \times 10^{-19} \times 1.05 \times 10^{-34} \times 1 \times 10^7} \right] = e^{-283} \approx 0 \]

The direct band-to-band tunneling probably is very low.

(c) To achieve 90\% modulation depth, \( e^{-2\alpha L} = 0.1 \).

Calculate \( L \) as a function of wavelength under electric field \( E = 2 \times 10^{3} \) V/cm. The result is plotted below.

4. (20\%) (a) Use the results for the optical density of states in bulk materials and absorption coefficient in thermal equilibrium, show that the total band-to-band spontaneous recombination rate of electron-hole pairs in a semiconductor with \( E_g \gg k_B T \) in thermal equilibrium at temperature \( T \) is \( R_{sp}^0 \approx \frac{2}{2\pi} \left( \frac{2\pi m_e^* k_B T}{\hbar^2} \right)^{3/2} e^{-E_g/k_B T} \).

(b) Use the result from (a) to show that the bimolecular recombination coefficient can be expressed by the following relation:
\[ B = \frac{1}{2\tau_{sp} \left( \frac{2\pi k_B T}{\hbar^2} \right)^{3/2} (m_e^* + m_h^*)^{3/2}}. \]

(a) \( \rho(\nu) = \frac{4\pi (2m_e^*)^{3/2}}{\hbar^2} (\nu - E_g)^{1/2} \)
\[
\alpha_0(v) = \frac{e^2}{8\pi n^2 v^2 \tau_{sp}} p(v)
\]

\[
R^0_{sp}(v) = \frac{8\pi n^2 v^2}{c^2} e^{\nu/k_B T} - 1 = \frac{4\pi(2m^*_0)^{3/2}}{h^2 \tau_{sp}} (h\nu - E_g) \frac{1/2}{1} \approx \frac{4\pi(2m^*_0)^{3/2}}{h^2 \tau_{sp}} (h\nu - E_g) \frac{1/2}{1} e^{-\nu/k_B T}
\]

for \( E = h\nu \geq E_g \gg k_B T \).

\[
R^0_{sp} = \int_0^\infty R^0_{sp}(v) d\nu
\]

\[
\approx \frac{4\pi(2m^*_0)^{3/2}}{h^2 \tau_{sp}} \int_{E_g}^\infty (E - E_g) \frac{1/2}{1} e^{-\nu/k_B T} dE = \frac{4\pi(2m^*_0 k_B T)^{3/2}}{h^2 \tau_{sp}} e^{-E_g/k_B T} \int_{0}^{\infty} x^{1/2} e^{-x} dx
\]

\[x = \frac{E - E_g}{k_B T}, \int_{0}^{\infty} x^{1/2} e^{-x} dx = \frac{\sqrt{\pi}}{2}
\]

\[
R^0_{sp} = \frac{2}{\tau_{sp}} \left( \frac{2m^*_0 k_B T}{h^2} \right)^{3/2} e^{-E_g/k_B T}
\]

(b) \( R^0_{sp} = Bn_0 P_0 = B n'_0 = B \cdot 4 \left( \frac{2\pi k_B T}{h^2} \right)^3 \left( m^*_0 \right)^{3/2} e^{-E_g/k_B T} \)

\[B = \frac{1}{2\tau_{sp}} \left( \frac{2\pi k_B T}{h^2} \right)^{-3/2} \left( m^*_0 + m^*_h \right)^{-3/2} = \frac{1}{2\tau_{sp}} \left( \frac{2\pi k_B T}{h^2} \right)^{-3/2} \left( m^*_e + m^*_h \right)^{-3/2}
\]

5. (20%) An AlGaInP/GaP LED emits at 607 nm wavelength. At this wavelength, the photopic luminous efficiency function has a value of \( V(\lambda) = 0.54113 \). Injection of 10 mA achieves a separation of quasi Fermi levels equal to \( E_g = 0.224 \) eV. The active volume of the LED is \( 10^5 \mu m^3 \). The LED is operated at a forward bias of 2 V. For simplicity, assumes the reduced effective mass \( m_r = 0.0594 m_0 \) and the spontaneous time constant \( \tau_{sp} = 500 \) ps.

(a) Calculate the total output photon flux.

(b) Assume all output photons are collected, calculate the external quantum efficiency, responsivity, and power conversion efficiency of the LED.

(c) Find its luminous efficiency and luminous flux.

(a) Total photon flux: \( \Phi_{out} = \frac{V}{\sqrt{2\hbar^3 \tau_{sp}}} \left( \frac{m^*_e}{\pi} \right)^{3/2} \left( k_B T \right)^{3/2} \exp \left( \frac{E_{Fc} - E_{Fv} - E_g}{k_B T} \right) \) sec\(^{-1}\)

\[
\Phi_{out} = \frac{10^5 \cdot 10^{-18}}{\sqrt{2(1.05 \times 10^{-34})^3 \cdot 500 \times 10^{-12}}} \left( \frac{0.0594 \cdot 9.1 \times 10^{-31}}{\pi} \right)^{3/2} \left( 0.026 \cdot 1.6 \times 10^{-19} \right)^{3/2} \exp \left( \frac{1.2 - 1.424}{0.026} \right)
\]

\[= 7.4 \times 10^{19} e^{-8.6} = 1.34 \times 10^{16} \) sec\(^{-1}\)

(b) External quantum efficiency: \( \eta_e = \frac{\Phi_{out}}{I / e} = \frac{1.34 \times 10^{16}}{10^{-2} / 1.6 \times 10^{-19}} = 21.4\%\)
Responsivity: \( R = \eta_v \frac{hv}{e} = 0.214 \times \frac{1.24}{0.607} = 0.437 \text{ W/A} \)

Power conversion efficiency: \( \eta_e = \eta_v \frac{hv}{eV} = \frac{0.437}{2} = 21.9\% \)

(c) Luminous efficiency: \( \eta_l = K \eta_v V(\lambda_0) = 683 \times 0.219 \times 0.54113 = 80.9 \text{ lm W}^{-1} \)

Luminous flux: \( \Phi_l = \eta_l P_p = 80.9 \times (0.01 \times 2) = 1.62 \text{ lm} \)