A Submodular Optimization Approach to Leader-Follower Consensus in Networks with Negative Edges

Andrew Clark, Qiqiang Hou, Linda Bushnell, and Radha Poovendran

***Abstract*— Networked systems often contain both positive and negative interactions between nodes, with the latter often represented as negative edge weights in a graph. Such negative edges may prevent a network from performing cooperative tasks such as achieving consensus, creating a need for new control-theoretic techniques that guarantee performance in the presence of negative edges. This paper considers the problem of selecting leader nodes to maintain a fixed state in order to steer the remaining nodes to a desired state value in networks with negative edges. We present two sufficient conditions that are equivalent to submodular constraints on the set of leader nodes. The first constraint is based on the graph spectrum, while the second is formulated in terms of the determinant of the Laplacian matrix. We prove that both conditions can be formulated as submodular constraints on the set of leader nodes, leading to polynomial-time algorithms with provable approximation guarantees for selecting a minimum-size set of leader nodes to satisfy these conditions. Furthermore, we introduce necessary conditions for consensus and prove that a set of leader nodes satisfying these conditions can be selected in polynomial time. We characterize the requirements for leader selection in order to ensure consensus in line networks. The results are illustrated through numerical study.**

I. Introduction

The growing importance of networked systems in applications such as energy, medicine, and social media has led to increased research attention into control of networks. A common approach to controlling networked systems is to directly control a subset of leader nodes, which influence the remaining (follower) nodes through local interaction rules. These interaction rules are defined by cooperative control algorithms, such as the consensus protocol, as well as natural or man-made coupling, including coupled oscillations [1], social opinion dynamics [2], and gene interactions [3]. Recent efforts have focused on selecting input nodes to satisfy criteria such as robustness to noise, synchronization, and smooth convergence [4].

Distributed control algorithms typically assume that all nodes cooperate to achieve a common goal in coordination with their neighbors. Many real-world systems, however, contain antagonistic interactions between neighboring nodes. Social networks, for example, have disagreeing users that place negative weight on each others' opinions. Biological systems often have negative regulatory links, as when one gene inhibits another. Such antagonistic interactions are often modeled as negative-weighted edges in the underlying graph.

A challenge of controlling networks with negative edges is that standard results in multi-agent control, such as consensus of networks with positive weights, are no longer applicable. Negative edges can be viewed as causing nodes in the graph to repel each other, hence causing the state values of the nodes to converge and preventing consensus from taking place.

These challenges have motivated research into consensus in networks with antagonistic interactions. Recent efforts have derived necessary and sufficient conditions for signed consensus using electric circuits [5] and passivity analysis [6]. These works focus on signed consensus in networks without leaders, however, and have conditions that are applicable for special cases of networks, such as networks with only a single negative edge [5], [7] and networks where no cycle contains two negative edges [6]. A related but distinct problem, bipartite synchronization, has the goal of guaranteeing that a network with negative edges can be partitioned into two subgraphs with different state values in steady-state [8], [9]. At present, there are few conditions for consensus with negative edges in networks with leader nodes, let alone efficient algorithms for selecting leaders that guarantee consensus in antagonistic networks.

This paper proposes a submodular optimization framework for leader selection in networks with negative edges. The proposed approach is applicable to arbitrary graphs, including graphs with cycles containing multiple negative edges. We first derive two new sufficient conditions for consensus in networks with negative edges. The first condition is based on a bound on the graph spectrum, while the second condition is expressed in terms of the trace and determinant of the Laplacian matrix. We prove that both conditions are equivalent to submodular constraints on the set of leader nodes, and derive efficient algorithms with provable optimality bounds by exploiting submodularity. In addition to ensuring consensus, we consider the problem of guaranteeing that consensus is achieved with a desired convergence rate, and prove that this problem can be formulated and approximately solved within the same framework.

We then propose necessary conditions for consensus based on the span of the incidence vectors of the negative edges, and prove that a minimum-size leader set that satisfies the necessary conditions can be computed in polynomial time. We present necessary and sufficient conditions for consensus in line networks with negative edges. Our results are demonstrated through numerical study.

A. Clark and Q. Hou are with the Department of Electrical and Computer Engineering, Worcester Polytechnic Institute, Worcester, MA 01609 USA. aclark, qhou@wpi.edu

L. Bushnell and R. Poovendran are with the Department of Electrical Engineering, University of Washington, Seattle, WA 98195 USA. {lb2,rp3}@uw.edu

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The paper is organized as follows. In Section II, we state the system model and preliminaries. In Section III, we present a submodular optimization approach to ensuring consensus in networks with negative edges. In Section IV, we develop necessary conditions for signed consensus. Consensus in a line network is analyzed in Section V. Section VI contains numerical results. Section VII concludes the paper.

II. Model and Preliminaries

A. Graph Theory

A graph G consists of a set of nodes V and a set of edges $E \subseteq V \times V$. Let $n = |V|$. For each node i, the set $N(i) = \{j : (i,j) \in E\}$ is denoted the *neighbor set* of i. The number of neighbors $|N(i)|$ is the degree of node i and is denoted d_i. Each edge (i,j) has a real-valued weight W_{ij} associated with it. We assume throughout that the graph is undirected.

For each edge $e = (i,j) \in E$, we define an incidence vector $b_e \in \mathbb{R}^n$, with $(b_e)_i = 1$ and $(b_e)_j = -1$. The incidence matrix of a graph is the $n \times |E|$ matrix with column set $(b_e : e \in E)$. Abusing notation slightly, we write b_{ij} to denote the incidence vector corresponding to edge (i,j).

The Laplacian $n \times n$ matrix L for the graph G is defined by

$$L_{ij} = \begin{cases} -W_{ij}, & (i,j) \in E \\ \sum_{j \in N(i)} W_{ij}, & i = j \\ 0, & \text{else} \end{cases} \tag{1}$$

The Laplacian matrix can also be written as the weighted sum of the outer products of the incidence vectors, i.e.,

$$L = \sum_{(i,j) \in E} W_{ij} b_{ij} b_{ij}^T.$$

From this definition, we have that the Laplacian matrix of a graph with nonnegative edge weights is positive semidefinite.

We remark that multiple definitions of the Laplacian matrix have been proposed in the literature. In [10], the authors construct a Laplacian matrix in which the diagonal entry W_{ii} is equal to $\sum_{j \in N(i)} |W_{ij}|$. The definition of (1) is consistent with other related works such as [5], [6]. We plan to investigate input selection for the class of dynamics considered in [10] as future work.

As a further preliminary, for any finite set V, a function $f : 2^V \rightarrow \mathbb{R}$ is submodular if

$$f(S \cup \{v\}) - f(S) \geq f(T \cup \{v\}) - f(T)$$

for any $S \subseteq T$ and $v \notin T$. A function f is supermodular if $-f$ is submodular.

Finally, we use the notation $A \succ 0$ to denote that A is positive definite and $A \prec 0$ to denote that A is negative definite. We let $\lambda_{min}(A)$ and $\lambda_{max}(A)$ denote the minimum and maximum eigenvalues, respectively, of a symmetric matrix A.

B. System Model

We consider a network consisting of n distributed nodes, indexed in the set $V = \{1, \ldots, n\}$, with edge set E. Each node i has an associated state variable $x_i(t) \in \mathbb{R}$. There are two types of nodes in the network, namely leader nodes and follower nodes. The set of leader nodes is denoted S. The leader nodes have equal, constant states, i.e., there exists x_0 such that $x_i(t) \equiv x_0$ for all $i \in S$ and $t \geq 0$. The state dynamics of the follower nodes are defined by

$$\dot{x}_i(t) = -\sum_{j \in N(i)} W_{ij}(x_i(t) - x_j(t)), \tag{2}$$

where W_{ij} is the weight that node i assigns to inputs from node j. The dynamics (2) can be expressed in vector form as $\mathbf{x}(t) = -L\mathbf{x}(t)$, where L is the Laplacian matrix of the graph. We let L_{ff} denote the sub-matrix of L consisting of the rows and columns indexed in $V \setminus S$, i.e., the rows and columns corresponding to the follower nodes.

Let $E_+ = \{(i,j) \in E : W_{ij} > 0\}$ and $E_- = \{(i,j) \in E : W_{ij} < 0\}$. We decompose the matrix L as $L = L_+ - L_-$, defined by

$$L_+ = \sum_{(i,j) \in E_+} W_{ij} b_{ij} b_{ij}^T$$
$$L_- = \sum_{(i,j) \in E_-} |W_{ij}| b_{ij} b_{ij}^T$$

which are equal to the graph Laplacians of the subgraphs induced by the positive and negative edges, respectively.

We assume without loss of generality that $x_0 = 0$, so that all leader nodes have state values equal to 0. The goal of the system is for the followers to converge to the state of the leader nodes, and hence is equivalent to ensuring that $\lim_{t \rightarrow \infty} \mathbf{x}(t) = 0$. This condition holds if and only if the system $\dot{\mathbf{x}}_f(t) = -L_{ff}\mathbf{x}_f(t)$ is asymptotically stable. The following preliminary result provides an equivalent condition for asymptotic stability.

Lemma 1: Define the matrix $D(S)$ to be a diagonal matrix with $D(S)_{ii} = 1$ if $i \in S$ and $D(S)_{ii} = 0$ otherwise. The condition $L_{ff} \succ 0$ is equivalent to $L + \alpha D(S) \succ 0$ for α sufficiently large.

Proof: If $L + \alpha D(S) \succ 0$, then $L_{ff} \succ 0$, since L_{ff} is a submatrix of L. Now, suppose that $L_{ff} \succ 0$. The condition $L + \alpha D(S) \succ 0$ is equivalent to

$$\left(\begin{array}{cc} L_{ff} & L_{fl} \\ L_{lf} & L_{ll} + \alpha I_{|S| \times |S|} \end{array} \right) \succ 0.$$

By the Schur complement theorem, $L + \alpha D(S) \succ 0$ is therefore equivalent to $L_{ff} \succ 0$ (true by assumption) and $L_{ll} + \alpha I_{|S|\times|S|} - L_{lf} L_{ff}^{-1} L_{fl} \succ 0$. The matrix $L_{lf} L_{ff}^{-1} L_{fl}$ is a symmetric positive semidefinite matrix, and hence selecting α to be greater than or equal to the largest eigenvalue of $L_{lf} L_{ff}^{-1} L_{fl}$ is sufficient for positive definiteness of $L + \alpha D(S)$. ■

It is a known result that, if all edges are positive, asymptotic stability is satisfied, and hence consensus is achieved, iff each follower node is path-connected to at least one input node [11]. Asymptotic stability in the positive-edge case follows from the fact that $L \succ 0$. This result, however, does not necessarily hold in the presence of negative edges [5]. In the next section, we present sufficient conditions for consensus in networks with negative edges.

III. Problem Formulation and Leader Selection Algorithm

This section formulates the problem of selecting leader nodes to guarantee consensus in networks with negative edges. We first formulate sufficient conditions for consensus, and then map those conditions to a constraint on a supermodular function. Leader selection algorithms are derived based on this supermodular structure. We consider the problem of maximizing the rate of convergence in signed networks and derive an analogous submodular optimization approach. Finally, we formulate a condition for consensus based on the trace and determinant of the Laplacian matrix.

A. Problem Formulation

The following theorem gives a sufficient condition for a set of leader nodes to guarantee consensus. In what follows, let $\beta = |\lambda_{min}(L)|$.

Theorem 1: Let β denote the maximum eigenvalue of L_{-}. If there exists α such that

$$\mathbf{trace}((L_{+} + \alpha D(S))^{-1}) \leq \frac{1}{\beta}, \tag{3}$$

then the system achieves consensus.

Proof: The condition $L + \alpha D(S) \succ 0$ is equivalent to $L_{+} + \alpha D(S) \succ L_{-}$. A sufficient condition for this to hold is $\lambda_{min}(L_{+} + \alpha D(S)) > \beta$, or equivalently, if $\frac{1}{\lambda_{min}(L+\alpha D(S))} < \frac{1}{\beta}$.

Now, since $L_{+} + \alpha D(S)$ is a symmetric, positive definite matrix,

$$\frac{1}{\lambda_{min}(L_{+} + \alpha D(S))} = \lambda_{max}((L_{+} + \alpha D(S))^{-1}).$$

Furthermore, positive definiteness of $L + \alpha D(S)$ implies positive definiteness of $(L_{+} + \alpha D(S))^{-1}$, and hence

$$\begin{aligned} &\lambda_{max}((L_{+} + \alpha D(S))^{-1}) \\ &\leq \sum_{i=1}^{n} \lambda_i((L_{+} + \alpha D(S))^{-1}) \\ &= \mathbf{trace}((L_{+} + \alpha D(S))^{-1}). \end{aligned}$$

Thus (3) is a sufficient condition for consensus. ■

Define the function $F(S) = \mathbf{trace}((L_{+} + \alpha D(S))^{-1})$. The function $F(S)$ is related to the *effective resistance* of the graph [12], which is defined by $R(S) = \mathbf{trace}(L_{ff}^{-1})$ for graphs with positive edge weights, and is known to be supermodular as a function S [13]. This intuition leads to the following result.

Theorem 2: The function $F(S)$ is monotone nonincreasing and supermodular.

A proof can be found in the appendix. The problem of selecting a minimum-size set of leader nodes to ensure that the sufficient condition of Theorem 1 is satisfied can be formulated as

$$\begin{array}{ll} \text{minimize} & |S| \\ S \subseteq V & \\ \text{s.t.} & F(S) \leq \frac{1}{\beta} \end{array} \tag{4}$$

A submodular algorithm for selecting a set of leaders is defined in the next section.

B. Leader Selection Algorithm

Problem (4) can be approximated up to a provable optimality bound using a greedy algorithm, which is described as Algorithm 1.

Algorithm 1 Algorithm for selecting leader nodes to ensure signed consensus.

```
procedure NEGATIVE_CONSENSUS_SELECTION(G, W)
    Input: Graph G = (V, E), weight matrix W
    Output: Leader set S
    S ← ∅
    Compute bound β
    while F(S) ≤ Σ_{k=1}^{n} ε_k do
        v* ← arg min {F(S ∪ {v}) : v ∈ V \ S}
        S ← S ∪ {v*}
    end while
    return S
end procedure
```

In Algorithm 1, the leader set is initialized as empty. At each iteration, the node v that minimizes the objective function $F(S \cup \{v\})$ is selected and added to the leader set. The algorithm terminates when the constraint of (4) is satisfied. The optimality bound of Algorithm 1 is as follows.

Proposition 1: Let S^* denote the optimal solution to (4). The set S selected by Algorithm 1 satisfies

$$\frac{|S|}{|S^*|} \leq 1 + \ln\left\{\frac{F(\emptyset) - \beta}{F(\hat{S}) - \beta}\right\}, \tag{5}$$

where $\hat{S}$ is the set obtained at the second-to-last iteration of Algorithm 1.

Proof: The proof follows from [14] and the supermodularity of $F(S)$ by Theorem 2. ■

C. Maximizing Convergence Rate

In addition to ensuring that consensus is achieved, maximizing the rate of convergence to consensus will minimize performance degradation due to errors in intermediate state values. In this section, we incorporate rate of convergence into the proposed framework for signed consensus. We first state a well-known preliminary result.

Lemma 2 ([15]): Suppose that there exists a positive definite function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\dot{V}(z) \leq -\alpha V(z)$ for some $\alpha > 0$. Then there exists a constant M such that $||z(t)||_2 \leq Me^{-\alpha t/2}||z(0)||$.

From Lemma 2, we derive the following result on ensuring that consensus is achieved at a desired rate.

Proposition 2: Suppose that $F(S) \leq \frac{1}{\beta+\gamma}$. If $x_i(0) = 0$ for all $i \in S$, then $||\mathbf{x}(t)||_2 \leq Me^{-\gamma t/2}||\mathbf{x}(0)||_2$ for some $M \geq 0$.

Proof: Define $V(x) = \frac{1}{2}x^T x$. It suffices to show that $\dot{V}(x) \leq -\gamma x^T x$, which is equivalent to $x^T L x \geq \gamma x^T x$. Furthermore, when $x_i(t) \equiv 0$ for $i \in S$, $x^T L x = (x^T(L + \alpha D(S))x)$, and hence a sufficient condition is $(L+\alpha D(S)+ betaI) \succ (\beta+\gamma)I$. A similar analysis to Theorem 1 implies that $\mathbf{trace}((L+\alpha D(S)+\beta I)^{-1}) \leq \frac{1}{\beta+\gamma}$ is sufficient, which is equivalent to $F(S) \leq \frac{1}{\beta+\gamma}$. ■

Algorithm 1 can be modified to ensure a desired convergence rate by substituting $\frac{1}{\beta+\gamma}$ instead of β into Line 6 of Algorithm 1.

D. Trace and Determinant Based Bounds

An alternative bound with submodular structure can also be derived using the trace and determinant of the matrix $(L + \alpha D(S) + \beta I)$. We first have a preliminary result.

Lemma 3 ([16]): Let A be a symmetric positive definite matrix. Then the minimum eigenvalue of A satisfies

$$\lambda_{min}(A) \geq \left(\frac{n-1}{\mathbf{trace}(A)}\right)^{n-1} \det(A).$$

Hence, a sufficient condition for consensus is to ensure that

$$\left(\frac{n-1}{\mathbf{trace}(L+\alpha D(S)+\beta I)}\right)^{n-1} \det(L+\alpha D(S)+\beta I) \geq \beta,$$

or equivalently

$$\log\det(L+\alpha D(S)+\beta I) - (n-1)\log(\mathbf{trace}(L)+\alpha|S|+\beta n) > \log\beta - (n-1)\log(n-1). \quad (6)$$

The following lemma provides some structure that will be used to develop algorithms for ensuring that this constraint is satisfied.

Lemma 4: The functions

$$f_1(S) \triangleq \log(\mathbf{trace}(L) + \alpha|S| + \beta n)$$
$$f_2(S) \triangleq \log\det(L + \alpha D(S) + \beta I)$$

are submodular as functions of S.

A proof can be found in the appendix. In general, maximizing the difference of two submodular functions is a computationally difficult problem [17]. As an alternative, we propose Algorithm 2, which uses the greedy algorithm as a subroutine to achieve the bound of (6). Define the function

$$G(S) = \log\det(L + \alpha D(S) + \beta I).$$

For each value of k, Algorithm 2 solves a greedy submodular maximization problem on $G(S)$ that satisfies a bound analogous to Proposition 1. By adding γ to the right-hand side of (6), consensus at rate γ can be ensured as well.

Algorithm 2 Algorithm for selecting leader nodes to ensure signed consensus.

procedure NEGATIVE_CONSENSUS_DETERMINANT(G, W)
  Input: Graph $G = (V, E)$, weight matrix W
  Output: Leader set S
  $S \leftarrow \emptyset$
  Compute bound β
  $r \leftarrow \mathbf{trace}(L)$
  for $k = 1, \ldots, n$ **do**
    $\epsilon \leftarrow (n-1)\log(r + \alpha k + \beta n) + \log\beta - (n-1)\log(n-1)$
    while $G(S) \leq \epsilon$ **do**
      $v^* \leftarrow \arg\max\{G(S \cup \{v\}) : v \in V \setminus S\}$
      $S \leftarrow S \cup \{v^*\}$
    end while
    if $|S| \leq k$ **then**
      return S
    end if
  end for
  return V
end procedure

IV. NECESSARY CONDITIONS FOR CONSENSUS

This section presents necessary conditions for consensus, as well as a procedure for selecting leader nodes that satisfy the necessary conditions. The following lemma gives a simple necessary condition for consensus to occur.

Lemma 5: Let $e = (i, j)$ denote an edge with negative weight w, and let b denote the corresponding incidence vector. If $b^T L^+ b < 2|w|$, then at least one of node i or node j must act as a leader in order for consensus to be achieved.

Proof: Consensus is guaranteed if and only if $x^T(L + \alpha D(S))x \geq 0$ for all x, or equivalently $x^T(L^+ + \alpha D(S))x \geq x^T L^- x$ for all x. The Laplacian L^- can be written as

$$L^- = \sum_{(i,j)\in E^-} |W_{ij}| b_{ij} b_{ij}^T.$$

Hence, if $b^T L^+ b < 2w$, then

$$b^T L^+ b < 2w \leq b^T L^- b,$$

implying that consensus cannot hold unless $b^T D(S) b > 0$. We have that $b^T D(S) b > 0$ iff $\{i, j\} \cap S \neq \emptyset$. ■

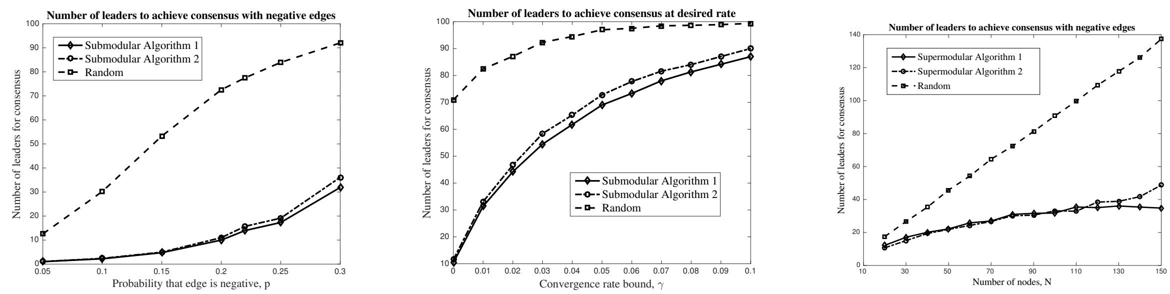
These conditions can also be generalized to vectors other than incidence vectors of negative edges. Let $v_1, \ldots, v_m$ denote the eigenvectors of L corresponding to negative eigenvalues. If the network reaches consensus, then

$$\text{span}(\{e_i : i \notin S\}) \cap \text{span}(\{v_1, \ldots, v_m\}) = \{0\}.$$

To see this, suppose that there exists a nonzero $x \in \text{span}(\{e_i : i \notin S\}) \cap \text{span}(\{v_1, \ldots, v_m\})$. Then

$$x^T(L + \alpha D(S))x = x^T L x < 0,$$

implying that $L + \alpha D(S) \prec 0$ and consensus is not achieved. A set of leader nodes can be selected to satisfy this



(a) (b) (c)

Fig. 1: Numerical study of leader selection for consensus in networks with negative edges, in a network of $N = 100$ nodes with range $r = 300$m. (a) Number of leaders required as a function of the fraction of negative edges, comparing the two submodular optimization algorithms (Algorithms 1 and 2) with random leader selection. The submodular algorithms require fewer leader nodes. (b) Leader selection to achieve a given bound γ on the rate of convergence. As the convergence rate bound increases, the number of leaders required by all algorithms increases, with the submodular algorithms consistently requiring fewer leaders than random selection. (c) Number of leaders with varying network size, $p = 0.3$. The number of leaders is roughly constant for the submodular algorithms.

more general condition as follows. Define the sets $A(T) = \mathrm{span}(\{e_i : i \in T\})$ and $Z = \mathrm{span}(\{v_1, \dots, v_m\})$, so that the goal is to ensure that $\dim(A(V \setminus S) \cap Z) = 0$. Equivalently,

$$\begin{aligned}\dim(A(V \setminus S)) + \dim(Z) - \mathrm{rank}([A(V \setminus S)\ Z]) \\ = |V \setminus S| + m - \mathrm{rank}([A(V \setminus S)\ Z]) = 0\end{aligned}$$

The condition $n - |S| + m - \mathrm{rank}([A(V \setminus S)\ Z]) = 0$ is equivalent to selecting a set T with maximum cardinality such that $\mathrm{rank}([A(T)\ Z]) = |T \cup Z|$. This, in turn, is equivalent to selecting a maximal set of vectors $T \subseteq \{e_1, \dots, e_n\}$ such that $[T\ Z]$ has full rank, which can be computed in polynomial time.

V. Consensus in Line Networks

We now present an example of leader-follower consensus with negative edges for a line graph. In this graph, the topology is defined by the edge set $E = \{(i, i+1) : i = 1, \dots, (n-1)\}$. We assume that all weights are equal to one. We have the following necessary condition for consensus.

Lemma 6: Suppose that, in the line graph, $(k, k+1)$ and $(l, l+1)$ are negative edges with $k < l$. If $\{k+1, \dots, l\} \cap S = \emptyset$, then consensus cannot be guaranteed.

Proof: Consider the vector $\mathbf{x} \in \mathbb{R}^n$ defined by $x_i = 1$ for $i \in \{k+1, \dots, l\}$ and $x_i = 0$ otherwise. Then $\mathbf{x}^T L \mathbf{x} = -2 < 0$, implying that L is not positive definite and there exist initial states such that consensus does not take place. ■

By Lemma 6, it suffices to consider a line network with $S = \{1, n\}$ and a single negative edge $(k, k+1)$. The following proposition describes consensus for this special case.

Proposition 3: Consensus is achieved in the line network with $S = \{1, n\}$ and single negative edge $(k, k+1)$ iff $n \leq 4$.

Proof: Suppose that $x_{k+1} - x_k = \alpha$. The minimum value of $x^T L x$ is given by

$$\begin{aligned}x^T L^+ x = x_2^2 + \sum_{i=2}^{k-1} (x_{i+1} - x_i)^2 + (x_{k+2} - (x_k + \alpha))^2 \\ + \sum_{i=k+2}^{n-2} (x_{i+1} - x_i)^2 + x_{n-1}^2.\end{aligned}$$

Computing the global minimizer of $x^T L^+ x$ yields $x_{n-1} = -x_2$, $x_{k+2} - x_k - \alpha = x_3 - x_2 = \cdots = x_{n-1} - x_{n-2} \triangleq \delta$. Furthermore, we have

$$x_{n-1} - x_2 = \sum_{i=2}^{n-2} (x_{i+1} - x_i) = \delta(n-4) + \alpha.$$

Finally, we have that $x_2 = \delta$, which yields $\delta = -\frac{\alpha}{n-2}$. Hence

$$x^T L^+ x = n \frac{\alpha^2}{(n-2)^2},$$

$x^T L^- x = \alpha^2$, and we have that consensus is guaranteed iff $\frac{n}{(n-2)^2} \geq 1$. Rearranging terms implies that this occurs when $n \geq 4$. ■

In a line topology, Proposition 3 implies that consensus occurs with input set $S = \{i_1, \dots, i_m\}$, with $i_1 < \cdots < i_m$, iff $|\{i_j + 1, \dots, i_{j+1}\}| \leq 9$ and the subgraph induced by $\{i_j + 1, \dots, i_{j+1}\}$ contains at most one negative edge.

VI. Numerical Study

A numerical study was conducted using Matlab. A network with $N = 100$ nodes was generated. The nodes were placed at random locations in a 1000×1000 square meter area. Each node had a range of 300 meters, so that an edge (i, j) was formed iff the two nodes were within 300 meters of each other. Links were chosen to be negative with probability p, where p ranged from 0.05 to 0.5. The edge weight magnitudes were chosen to be equal to the distance between the nodes.

We compared four algorithms, including the two submodular optimization algorithms (Algorithms 1 and 2). We also considered uniform random selection of leader nodes and choosing all nodes with negative edges as leaders.

The results of selecting a minimum-size leader set for consensus are shown in Figure 1(a). When the number of negative edges is relatively small (only 5% of the total number of edges), the submodular algorithms only require a single leader node on average, while selecting random nodes requires ten leaders on average. As the fraction of negative edges increases, the number of leaders required by the submodular algorithms increases to roughly twenty, while nearly all nodes must be selected as leaders under the random heuristic.

Leader selection to achieve a desired bound on convergence rate is shown in Figure 1(b), for a network where $p = 0.2$. The number of leaders required to achieve the convergence rate γ is increasing in γ, with both submodular algorithms requiring fewer leaders than random selection. In this case, Algorithm 1 selects fewer leader nodes to achieve the same convergence rate bound than Algorithm 2.

Figure 1(c) shows the number of leaders needed as a function of the network size N when the probability of a negative edge $p = 0.3$. A constant fraction of leaders is needed for the random selection algorithm, while the number of leaders is roughly constant for the submodular algorithms.

VII. Conclusion

Networked systems typically contain both positive and negative (antagonistic) interactions between nodes. Negative interactions may impede cooperative tasks such as consensus, creating a need for new control strategies that mitigate these challenges. This paper considered the problem of ensuring consensus in networks with negative edges by pinning a subset of leader nodes to a fixed value. We formulated both sufficient and necessary conditions for consensus in such networks. We proposed two sufficient conditions, which we mapped to submodular constraints on the set of leader nodes, and developed efficient algorithms for approximating the minimum-size set of nodes to satisfy the conditions. Furthermore, we extended both conditions to arrive at bounds on the rate at which consensus is achieved. We also developed rank-based necessary conditions for consensus that can be guaranteed using polynomial-time algorithms. We then analyzed consensus in line networks with negative edges and derived necessary and sufficient conditions for a set of leader nodes to guarantee consensus in such networks. Our approach was evaluated through numerical study.

Consensus in networks with antagonistic interactions is currently an active research area. Future works will aim to develop tighter necessary and sufficient conditions that can be ensured using polynomial-time algorithms. Efficient control strategies with time-varying leader states, as opposed to fixed leader states, will also be considered. Finally, consensus in directed networks with negative edges and switching topologies will be explored in future work.

Appendix

We now present proofs of Theorem 2 and Lemma 4.

Proof: [Proof of Theorem 2] Let $S \subseteq T$ and suppose $v \notin T$. Define $X = L_{+} + \alpha D(S)$ and $Y = L_{+} + \alpha D(T)$, and let $f : S_{++}^{n} \rightarrow \mathbb{R}$ be defined by $f(X) = \mathbf{tr}(X^{-1})$. Then submodularity of $F(S)$ is equivalent to

$$f(X) - f(X + \alpha e_v e_v^T) \geq f(Y) - f(Y + \alpha e_v e_v^T).$$

In order to demonstrate this, it suffices to show that for $B = \alpha(D(T) - D(S))$, the function $\overline{f}(t) = f(X + tB) - f(X + tB + \alpha e_i e_i^T)$ is decreasing. Using standard matrix derivatives yields

$$\begin{aligned}&\overline{f}'(t) = -\mathbf{trace}((X + tB)^{-1}B(X + tB)^{-1})\\&+ \mathbf{trace}((X + tB + \alpha e_i e_i^T)^{-1}B(X + tB + \alpha e_i e_i^T)^{-1})\end{aligned}$$

and hence it suffices to show that

$$\mathbf{trace}((X + tB + \alpha e_i e_i^T)^{-2}B) \leq \mathbf{trace}((X + tB)^{-2}B).$$

Similarly to the above, define $Z = \alpha e_i e_i^T$, so that the goal is to show that the function

$$g(u) = \mathbf{trace}((X + tB + uZ)^{-2}B)$$

is decreasing in u. We have that

$$\begin{aligned}\frac{dg}{du} &= \mathbf{trace}((-(X + tB + uZ)^{-2}Z(X + tB + uZ)^{-1}\\&\quad -(X + tB + uZ)^{-1}Z(X + tB + uZ)^{-2})B)\\&= -\mathbf{trace}(((X + tB + uZ)^{-2}Y(X + tB + uZ)^{-1}\\&\quad +(X + tB + uZ)^{-1}Y(X + tB + uZ)^{-2})B)\\&\leq 0.\end{aligned}$$

The final inequality follows from the fact that $(X+tB+uZ)$ is an inverse positive matrix (since it is a positive definite matrix with negative off-diagonal entries) and B has all positive entries, and hence the trace of products of $(X + bB + uZ)^{-1}$ and B is positive. ∎

Proof: [Proof of Lemma 4] The function $f_1(S)$ has the form $f_1(S) = \tilde{f}(|S|)$, where $\tilde{f} : \mathbb{R} \rightarrow \mathbb{R}$ is the increasing concave function $\tilde{f}(x) = \log(\mathbf{trace}(L) + \alpha x + \beta n)$. Since $f_1(S)$ is equal to a concave function of the cardinality of S, it is concave as a function of S.

The function $f_2(S)$ is submodular by [18]. ∎

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