Input Selection for Disturbance Rejection in Networked Cyber-Physical Systems

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***Abstract*— The ability to maintain functionality in the presence of disturbances is a critical requirement of networked systems. A system satisfies the disturbance decoupling property if, for any additive disturbance signal, there exists a control input such that the system output resulting from the combined disturbance and control signals is equivalent to the output in the absence of any disturbance. In this paper, we present a submodular optimization approach for selecting input nodes to guarantee that the disturbance decoupling property is satisfied. Our approach is based on mapping three known criteria for disturbance decoupling to two sufficient matroid constraints. This mapping implies that the problem of selecting a minimum-size sufficient input set for disturbance decoupling, as well as the problem of selecting a minimum-cost input set, can be solved in polynomial time as matroid intersection problems. Our results are illustrated via simulation study.**

I. Introduction

Networked cyber-physical systems (CPS) play a vital role in health, transportation, and energy infrastructure. One widely-implemented approach to controlling CPS, in order to guarantee robustness, performance, and stability, is to directly control a subset of input nodes. The input nodes then influence the remaining nodes via local interconnections, thus providing an input signal that propagates through the system [1].

The importance and complexity of networked CPS leaves them prone to disturbances, including measurement noise, errors in actuation, model uncertainties, and adversarial attacks. Such systems must be designed to maintain stability and provide performance guarantees even in the presence of such disturbances. While robustness to disturbances can be enhanced through H_∞ control [2], the robust controller will inherently depend upon which nodes are chosen to act as inputs. Hence, a design problem is to select a set of input nodes such that a disturbance-rejecting controller can be synthesized, viewing the states of the input nodes as control variables. Selecting such a subset of nodes is a combinatorial problem, and is therefore complementary to existing control-theoretic techniques for ensuring disturbance rejection.

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Disturbance decoupling and pole placement have been identified as two key criteria for ensuring stability and performance in the presence of disturbances [3]. The disturbance decoupling property is satisfied if, for any disturbance signal, a control input signal can be chosen such that the output of the system is the same as the output without any disturbance. In essence, disturbance decoupling implies that an input can be chosen that cancels out the impact of the disturbance. Pole placement refers to the ability to place the system poles at arbitrary points in the complex plane, thus ensuring that a desired stability margin is satisfied. Both system properties depend on the choice of input nodes.

Necessary and sufficient conditions for disturbance decoupling and pole placement (DDPP) with known and fixed inputs have been proposed in the control-theoretic literature [3], [4]. In [4], necessary and sufficient conditions for DDPP that depend only on the structure of the system, and guarantee DDPP for almost any system parameters, are derived for systems with given and fixed inputs. Currently, however, there are no analytical approaches to selecting input nodes that guarantee DDPP.

In this paper, we present a matroid optimization approach to selecting a minimum-size subset of input nodes in order to guarantee DDPP. Our approach is based on mapping the criteria identified in [4] to two matroid constraints, which can be interpreted via a bipartite graph induced the network dynamics. The first matroid constraint states that there exists a maximum matching in which all non-input nodes are matched to nodes that are not impacted by disturbances. The second matroid constraint implies that there exists a maximum matching in which no node is matched to itself.

Taken together, selecting an input set that satisfies the two derived conditions can be viewed as finding a maximum-cardinality intersection between two matroids, which is known to be solvable in polynomial time [5]. We present a polynomial-time algorithm for selecting a minimum-size set of input nodes to satisfy the sufficient conditions for DDPP, as well as a generalized problem in which a minimum-cost subset of input nodes is selected. We compare the performance of the matroid intersection approach with greedy, degree-based, and random heuristics via numerical study, and find that the matroid intersection algorithm requires the same number of inputs as the greedy algorithm and significantly fewer inputs than the degree-based and random selection algorithms.

The paper is organized as follows. Section II provides an overview of the related work. The system model and background on disturbance decoupling are given in Section

III. Section IV contains the problem formulation. Input selection algorithms for disturbance decoupling and pole placement are given in Section V. Section VII contains simulation results. Section VIII concludes the paper.

II. Related Work

Necessary and sufficient conditions for disturbance decoupling in systems with given parameters were presented in [3]. In [4], conditions for generic disturbance decoupling (i.e., disturbance decoupling for all but a measure-zero set of parameter values) and pole placement in structured systems were derived, based on graph matchings.

More recently, disturbance decoupling was proposed as a framework for robust formation control in [6]. Graph-based disturbance decoupling conditions were presented in [7]. The design of networked control systems that achieve disturbance decoupling with a minimum number of links was studied in [8]. These works, however, assume that the control inputs are given and fixed, and do not provide efficient algorithms for selecting input nodes to ensure disturbance decoupling.

Optimal selection of input nodes in networked system has received recent research interest. In [1], a polynomial-time algorithm for selecting the minimum-size set of input nodes to guarantee controllability was presented. Subsequent works incorporated features including controller energy [9], [10] and joint optimization of performance criteria [11]. Controllability, however, is necessary but not sufficient for disturbance decoupling [3], and hence these existing works do not guarantee the disturbance decoupling property. In [12], [13], techniques were proposed for selecting input nodes to minimize errors due to Gaussian noise. These techniques are applicable to Gaussian noise and do not consider arbitrary disturbances as in this paper.

III. Model and Preliminaries

In this section, we present the system model considered and definition of DDPP. We provide background on matroids and graph matchings, followed by matching-based conditions for DDPP.

A. System Model

We consider a system consisting of n state variables, represented by a vector-valued state $\mathbf{x}(t) \in \mathbb{R}^n$. The set of input signals is denoted $\mathbf{u}(t) \in \mathbb{R}^m$, while the disturbance is defined by $\mathbf{d}(t) \in \mathbb{R}^l$. The system output is denoted $\mathbf{y}(t) \in \mathbb{R}^k$. The state dynamics (Σ) are defined by

$$(\Sigma) \left\{ \begin{array}{l} \dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) + Q\mathbf{d}(t) \\ \mathbf{y}(t) = H\mathbf{x}(t) \end{array} \right. \quad (1)$$

The matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $Q \in \mathbb{R}^{n \times l}$, and $H \in \mathbb{R}^{k \times n}$ are *structured*, meaning each entry is either zero or a free parameter that is unknown at the design phase. The structured systems assumption leads to DDPP conditions that hold for almost any choice of the free parameters (i.e., all free parameter values except for a set of Lebesgue measure zero), and hence are not sensitive to uncertainties in the state dynamics.

We further assume that the matrices B, H, and Q contain exactly one nonzero (free) entry per row and column, which models the case where nodes receive independent inputs and disturbances, and the outputs are in the form of state feedback. Let R denote the set of nodes in $\{1, \ldots, n\}$ satisfying $Q_{ij} \neq 0$ for some $j \in \{1, \ldots, l\}$. The set of input nodes, denoted S, consists of the nodes $\{i : B_{ij} = 0 \text{ for some } j\}$. Define $T \triangleq V \setminus S$ to be the set of non-input nodes.

The vector $\mathbf{x}(t)$ can be interpreted as a vector of n node states, while the matrix A defines the network topology between the nodes. The conditions on the matrices B, H, and Q imply that nodes receive independent inputs or disturbances, and provide scaled state outputs.

The disturbance decoupling and pole placement property is defined as follows.

Definition 1 ([4]): The system (Σ) satisfies the disturbance decoupling and pole placement (DDPP) property if for any polynomial $p(s)$, we can construct matrices F and R such that $\mathbf{u}(t) = F\mathbf{x}(t) + R\mathbf{d}(t)$, and the conditions

$$H(sI - (A + BF))^{-1}(Q + BR) = 0 \quad (2)$$
$$\det(sI - (A + BF)) = p(s) \quad (3)$$

are satisfied.

Eq. (2) is the disturbance decoupling property; it can be obtained by setting $\mathbf{u}(t) = F\mathbf{x}(t)+R\mathbf{d}(t)$ in Eq. (1), yielding $\dot{\mathbf{x}}(t) = (A + BF)\mathbf{x}(t) + (Q + BR)\mathbf{d}(t)$. From here, (2) is equivalent to the transfer function from input $\mathbf{d}(t)$ to output $\mathbf{y}(t)$ being identically zero, i.e., the impact of the disturbance on the system output is negated by the control input.

Eq. (3) is the pole placement property, and implies that the control matrix F can be selected so that the poles of the transfer function have the values of the zeroes of $p(s)$. The following lemma characterizes the DDPP property in terms of solutions of polynomial matrix equations.

Lemma 1 ([4]): Define polynomial matrices $M(s)$ and $N(s)$ by

$$M(s) = \left(\begin{array}{cc} A - sI & B \\ H & 0 \end{array} \right), \quad N(s) = \left(\begin{array}{c} Q \\ 0 \end{array} \right).$$

Then (2) and (3) hold if and only if the matrix polynomial equation $M(s)X(s) = N(s)$ has both a polynomial solution $X_1(s)$ and a proper rational solution $X_2(s)$.

We will present equivalent graph-theoretic conditions for Lemma 1. As a preliminary, however, we require additional background in matching theory.

B. Background on Matching Theory

A bipartite graph $G = (V, W, E)$ consists of a set of nodes V, a set of nodes W with $W \cap V = \emptyset$, and a set of edges $E \subseteq W \times V$, so that all edges are between nodes in W and nodes in V. A *matching* on a bipartite graph is a set of edges $E' \subseteq E$ such that no two edges share a common vertex. A maximum matching is a matching with maximum cardinality over all matchings. If $|E'| = |V| = |W|$, then E' is a *perfect matching*.

A weight function $w : E \rightarrow \mathbb{R}$ can be assigned to the edges of the bipartite graph. A maximum-weight matching E' is a matching with $\sum_{e \in E'} w(e)$ maximized. A maximum-weight maximum matching is a matching E' satisfying

$$E' = \arg\max \left\{ \sum_{e \in E''} w(e) : E'' \text{ is maximum} \right\}.$$

For a weight function w, we define $\alpha_w(G)$ as the weight of a maximum-weight maximum matching with weight function w, and $\beta_w(G)$ as the weight of a minimum-weight maximum matching with weight function w.

Let E' be a matching in a bipartite graph. An alternating path is a path in which every other edge is contained in E'. Alternating paths can be used to construct graph matchings. Suppose that $w_1, w_2 \in W$ and there is an alternating path $e_1, e_1', e_2, e_2', \ldots, e_r, e_r'$ between w_1 and w_2, with $\{e_1', \ldots, e_r'\} \subseteq E'$. Then $E'' = E' \setminus \{e_1', \ldots, e_r'\} \cup \{e_1, \ldots, e_r\}$ defines a matching on G.

We now define the *Dulmage-Mendlesohn decomposition* [14], which provides a decomposition of maximum matchings on bipartite graphs that will be needed in the necessary and sufficient conditions for DDPP in subsequent sections. Define W_0 to be the set of nodes in W that are unmatched by at least one maximum matching on G, and define V_0 to be the set of neighbors of W_0. Similarly, let V_∞ denote the set of nodes in V that are unmatched by at least one maximum matching on G, and let W_∞ denote their neighbor set. Finally, define $W_* = W \setminus (W_0 \cup W_\infty)$ and $V_* = V \setminus (V_0 \cup V_\infty)$. The DM decomposition is equal to $V = V_0 \cup V_* \cup V_\infty$ and $W = W_0 \cup W_* \cup W_\infty$. The following result describes the structure of maximum matchings on G in terms of the DM decomposition.

Lemma 2 ([14]): Let E' be a maximum matching on a bipartite graph G. Then $E' = E_0' \cup E_*' \cup E_\infty'$, where E_0' is a maximum matching on (W_0, V_0) with cardinality $|V_0|$, E_*' is a perfect matching on (W_*, V_*), and E_∞' is a maximum matching on (W_∞, V_∞) with cardinality $|W_\infty|$.

Furthermore, the following lemma leads to efficient algorithms for computing the DM decomposition.

Lemma 3: Let E' be a maximum matching on a bipartite graph G. Then W_0 is equal to the set of nodes that are connected, via an alternating path, to a node that w is unmatched by E', while V_∞ is equal to the set of nodes that are connected, via an alternating path, to a node v that is unmatched by E'.

We note that the DM decomposition does not depend on the choice of maximum matching E' in Lemma 3.

C. Matching-Based Conditions for DDPP

We now present conditions for DDPP in structured systems based on graph matchings. All results of this subsection can be found in [4]. As a preliminary, let $T(s)$ be a polynomial matrix. Define $r(T(s))$ to be the maximum rank of $T(s)$ for any complex s. Let $\Lambda_r(T(s))$ denote the degree of the greatest common divisor of the r-th order minors of $T(s)$, and let $\Delta_r(T(s))$ denote the maximum degree of all r-th order minors of $T(s)$. The following proposition provides necessary and sufficient conditions for DDPP for almost any choice of the free parameters in A, B, C, and Q.

Proposition 1: Let $M(s)$ and $N(s)$ be defined as in Lemma 1. Then DDPP holds iff (i) $r(M(s)) = r([M(s)\ N(s)])$, (ii) $\Lambda_r(M(s)) = \Lambda_r([M(s)\ N(s)])$, and (iii) $\Delta_r(M(s)) = \Delta_r([M(s)\ N(s)])$.

The three conditions of Proposition 1 can each be mapped to equivalent matching conditions. We define a bipartite graph $G(T(s))$ for a polynomial matrix $T(s) \in \mathbb{R}^{a \times b}$ as follows. The bipartite graph has node sets V and W, where $|W| = b$ and represents the columns of $T(s)$, $|V| = a$ and represents the rows of $T(s)$, and an edge (i,j) exists from $i \in W$ to $j \in V$ if $[T(s)]_{ji} \neq 0$.

We define two weight functions ξ_+ and ξ_- on $G(T(s))$. For an edge (i,j), the weight function $\xi_+(i,j)$ is equal to the degree of $[T(s)]_{ji}$, while $\xi_-(i,j)$ is equal to the smallest exponent of s in $[T(s)]_{ji}$ with a nonzero coefficient. The following proposition relates the conditions of Proposition 1 to maximum matchings on the graph $G(T(s))$.

Proposition 2: Let $G(T(s)) = G_0(T(s)) \cup G_*(T(s)) \cup G_\infty(T(s))$ denote the DM decomposition of $G(T(s))$. The rank $r(T(s))$ is equal to the cardinality of a maximum matching on $G(T(s))$. $\Delta_r(T(s))$ is equal to $\alpha_{\xi_+}(G(T(s)))$. $\Lambda_r(T(s))$ is equal to $\beta_{\xi_-}(G_0(T(s))) + \alpha_{\xi_+}(G_*(T(s))) + \beta_{\xi_-}(G_\infty(T(s)))$.

Proposition 2 implies that the DDPP property can be checked for a system with given inputs, disturbances, and outputs in polynomial-time via matching-based conditions.

D. Background on Matroids

Matroids provide additional structure that enables efficient solution of otherwise intractable discrete optimization problems. In this section, we define the concept of a matroid and present two classes of matroids that will be used in our subsequent formulation.

Definition 2: Let V be a finite set and $\mathcal{I}$ be a set of subsets of V. The pair $\mathcal{M} = (V, \mathcal{I})$ defines a *matroid* if (M1) $\emptyset \in \mathcal{I}$, (M2) $B \in \mathcal{I}$ and $A \subseteq B$ implies $A \in \mathcal{I}$, and (M3) if $A, B \in \mathcal{I}$ and $|A| < |B|$, then there exists $v \in B \setminus A$ such that $(A \cup \{v\}) \in \mathcal{I}$.

Let $G = (V, W, E)$ be a bipartite graph, and define $\mathcal{I} = \{W' \subseteq W : W' \text{ is matched under a matching } E'\}$. The collection $\mathcal{M} = (W, \mathcal{I})$ defines a *transversal matroid*; a transversal matroid can be defined similarly using V.

IV. Problem Formulation - Input Selection for Disturbance Rejection

In this section, we formulate the problem of selecting the minimum-size set of input nodes S to satisfy the DDPP conditions of Proposition 1. We formulate a matroid constraint on S that is sufficient for (i) and (ii), followed by a matroid constraint that is sufficient for (iii). We then present the overall problem formulation of selecting a minimum-size input set for DDPP, which combines the two constraints.

A. Matroid Constraint for Conditions (i) and (ii)

We describe a sufficient condition to ensure that $r(M(s)) = r([M(s)\ N(s)])$, i.e., the rank is unchanged by adding columns corresponding to the disturbances. By Proposition 2, this is equivalent to the cardinality of the maximum matching on $G(M(s))$ being equal to the cardinality of the maximum matching on $G([M(s)\ N(s)])$. We first present a sufficient condition for ensuring that the maximum matchings have the same cardinality, and then prove that this sufficient condition can be represented as a transversal matroid constraint.

Some properties of the bipartite graph $G([M(s)\ N(s)])$ are described as follows. Since we assume that each row and column of Q contain exactly one nonzero entry, adding the columns $N(s)$ creates l additional nodes, which we denote $D = \{d_1, \ldots, d_l\}$, to the set W, each of which has a single edge to a distinct node in V. Similarly, the input nodes are represented by m additional nodes in W, denoted $u_1, \ldots, u_m$, each of which has a single edge to a distinct node in V. The outputs define a set $Y = \{y_1, \ldots, y_k\}$. Hence, the overall bipartite graph $G(M(s))$ is defined by $G(M(s)) = (W \cup U, V \cup Y, E)$, while the bipartite graph $G([M(s)\ N(s)]) = (W \cup U \cup D, V \cup Y, E)$. We let $G(S, D)$ denote the bipartite representation induced by input set S and disturbance set D. The following lemma gives a sufficient condition based on the DM decomposition of the bipartite graph representation.

Lemma 4: Suppose that, for each node $i \in R$, $v_i \in V_0(G(S, \emptyset))$. Then the cardinality of a maximum matching in $G(M(s))$ is equal to the cardinality of a maximum matching in $G([M(s)\ N(s)])$.

Proof: Let r and r' denote the cardinalities of maximum matchings in $G(M(s))$ and $G([M(s)\ N(s)])$, respectively. Clearly, $r \leq r'$. Suppose that $r' > r$, and let E' denote a maximum-cardinality matching on $G([M(s)\ N(s)])$. We must have that, for some $D' \subseteq D$, with $|D'| = r' - r$, $(d_i, v_i) \in E'$ for all $i \in D'$. Otherwise, by restricting E' to W, we would have a matching on $G(M(s))$ with cardinality greater than r, contradicting the definition of r.

Now, if we restrict E' to W, we have a matching of cardinality r in $G(M(s))$ with the nodes $\{v_i : i \in R'\}$ unmatched, where $R' = \{i : Q_{ij} \neq 0 \text{ for some } j \in D'\}$. Since $v_i \in V_0(G(S, \emptyset))$, however, each node v_i is matched under any maximum matching. This contradiction implies that $r' = r$. ■

Lemma 4 is also sufficient for condition (ii), as shown by the following theorem.

Theorem 1: Suppose that the conditions of Lemma 5 hold. Then the weight of the maximum ξ_+-weight maximum matching of $G(M(s))$ is equal to the weight of the maximum ξ_+-weight maximum matching of $G([M(s)\ N(s)])$.

Proof: By Lemma 2, any maximum matching can be decomposed into a maximum matching on G_∞, a perfect matching on G_*, and a maximum matching on G_0. By the conditions of Lemma 4, all of the disturbance nodes $d_1, \ldots, d_l$ lie in G_0. Hence the maximum ξ_+-weight of the maximum matching in G_∞ and G_* is unchanged, and it suffices to consider G_0.

Note that $\xi_+(w_i, v_j) = 1$ if and only if $i = j$, and hence the maximum-weight maximum matching corresponds to the maximum matching with the largest number of edges of the form (w_i, v_i). On the other hand, the output nodes $y_1, \ldots, y_k$ are disjoint from V_0. This is because, since each row and column of H contains exactly one nonzero entry, there is only a single edge incoming to each node y_i, corresponding to a node x_j. Hence x_j is matched in any maximum matching, since otherwise the matching could be increased by adding the edge (x_j, y_i). Thus either (x_j, y_i) is present in every maximum matching (in which case $y_i \in V_*$) or y_i is unmatched by some maximum matching (in which case $y_i \in V_\infty$).

We then have that V_0 consists entirely of nodes v_j corresponding to state entries. Furthermore, all neighbors of V_0 lie in W_0. Hence the matching $\{(w_i, v_i) : v_i \in V_0\}$ is a maximum matching on G_0, with total weight $|V_0|$, which is the largest possible weight of any maximum matching in G_0. Since this matching is valid in both $G_0(M(s))$ and $G_0([M(s)\ N(s)])$, the total maximum-weight maximum matching is preserved. ■

We now present a sufficient condition, which implies that $v_i \in V_0(G(S, \emptyset))$ for all $i \in D$.

Lemma 5: Define the bipartite graph $G' = (W \setminus D, V, E)$. If there is a perfect matching on G', then $v_i \in V_0(G(S, \emptyset))$ for all $i \in D$.

Proof: Suppose that E' is a perfect matching on G'. Then E' is a maximum matching on G with $\{w_i : i \in D\}$ unmatched, implying that $\{w_i : i \in D\} \subseteq W_0(S, \emptyset)$. Furthermore, since v_i is a neighbor of w_i for all $i = 1, \ldots, n$, and $w_i \in W_0(S, \emptyset)$ for all $i \in D$, we have that $v_i \in V_0(G(S, \emptyset))$. ■

A matroid condition that ensures $v_i \in V_0(G(S, \emptyset))$ for all $i \in D$, and hence is sufficient for (i) and (ii), is provided by the following theorem.

Theorem 2: Let $T = V \setminus S$. Define $\mathcal{M}_1 = (V, \mathcal{I})$ by $A \in \mathcal{I}$ iff there is a matching from $W \setminus R$ into $A \cup Y$ such that all vertices in $A \cup Y$ are matched. Then $\mathcal{M}_1$ is a transversal matroid and the conditions of Lemma 5 hold when $T \in \mathcal{M}_1$.

Proof: The fact that $\mathcal{M}_1$ is a transversal matroid follows from the definition of transversal matroid in Section III-D. Furthermore, if there is a matching E'' from $W \setminus D$ into $V \setminus S$, then this matching can be extended to a perfect matching E' on G' by setting $E' = E'' \cup \{(u_i, v_i) : i \in S\}$. Hence the conditions of Lemma 5 hold when $T \in \mathcal{M}_1$. ■

Theorems 1 and 2 imply that $T \in \mathcal{M}_1$ is sufficient for $r(M(s)) = r'([M(s)\ N(s)])$ and $\Delta_r(M(s)) = \Delta_r([M(s)\ N(s)])$, which correspond to conditions (i) and (ii) of Proposition 1.

B. Matroid Constraint for Condition (iii)

We now turn to condition (iii) of Proposition 1, which states that $\Lambda_r(M(s)) = \Lambda_r([M(s)\ N(s)])$, where $\Lambda_r(\cdot)$ is the sum of a minimum ξ_--weight maximum matching on

G_0, a maximum ξ_+-weight maximum matching on G_*, and a minimum ξ_--weight maximum matching on G_∞. We first prove that, if the conditions of Lemma 4 hold, then it suffices for the minimum ξ_--weight maximum matching on G to be unchanged by incorporating the disturbances.

Lemma 6: Suppose that $v_i \in V_0(G(S,\emptyset))$ for all $i \in R$. If the minimum ξ_--weight maximum matching on $G(M(s))$ has the same weight as the minimum ξ_--weight maximum matching on $G([M(s)\ N(s)])$, then $\Lambda_r(M(s)) = \Lambda_r([M(s)\ N(s)])$.

Proof: If $v_i \in V_0(G(S,\emptyset))$ for all $i \in R$, then the sets V_0, V_*, and V_∞ in the DM decomposition are the same in $G(M(s))$ and $G([M(s)\ N(s)])$. Hence, the minimum ξ_--weight maximum matching on G_0 and the maximum ξ_+-weight maximum matching on G_* are the same in $G(M(s))$ and $G([M(s)\ N(s)])$, since $G_0(M(s)) = G_0([M(s)\ N(s)])$ and $G_*(M(s)) = G_*([M(s)\ N(s)])$.

By Lemma 2, any maximum matching can be decomposed into a maximum matching on G_0, a perfect matching on G_*, and a maximum matching on G_∞. Since the minimum ξ_--weight maximum matchings on G_∞ and G_* are unchanged, if the minimum ξ_--weight maximum matching on G is unchanged, then the minimum ξ_--weight maximum matching on G_0 must be unchanged as well. Thus

$$\begin{aligned}\beta_{\xi_-}(G_0(M(s))) + \alpha_{\xi_+}(G_*(M(s))) + \beta_{\xi_-}(G_\infty(M(s)))\\ = \beta_{\xi_-}(G_0([M(s)\ N(s)])) + \alpha_{\xi_+}(G_*([M(s)\ N(s)]))\\ + \beta_{\xi_-}(G_\infty([M(s)\ N(s)])),\end{aligned}$$

implying $\Lambda_r(M(s)) = \Lambda_r([M(s)\ N(s)])$ by Proposition 2. ■

We further observe that, if the minimum ξ_--weight of a maximum matching on $G(M(s))$ is zero, then the minimum ξ_--weight maximum matching on $G([M(s)\ N(s)])$ will be zero as well, provided that the cardinality of a maximum matching on $G(M(s))$ and $G([M(s)\ N(s)])$ is the same. Hence if $\beta_{\xi_-}(G(M(s))) = 0$ and $v_i \in V_0G((S,\emptyset))$ for all $i \in R$, then condition (iii) will hold by Lemma 6. The following lemma gives a necessary and sufficient condition for $\beta_{\xi_-}(G(M(s))) = 0$.

Lemma 7: Let G' be a graph obtained from $G(M(s))$ by removing all edges (w_i, v_i) with $A_{ii} = 0$. Then the minimum ξ_--weight of a maximum matching on $G(M(s))$ is zero if and only if there is a matching of cardinality $r(M(s))$ in G'.

Proof: Note that the weight $\xi_-(w_i, v_j) = 1$ if and only if $j = i$ and $A_{ii} = 0$. Suppose that there is a matching E'' of cardinality $r(M(s))$ in G'. Then E'' is a maximum matching in $G(M(s))$, and since no edges (w_i, v_i) with $A_{ii} = 0$ are present in the matching, the total ξ_--weight of the matching is 0.

Suppose now that the minimum ξ_--weight of a maximum matching in $G(M(s))$ is zero. Let E'' be such a matching. Then there are no edges of the form (w_i, v_i) with $A_{ii} = 0$ in E'', and hence E'' is also a matching of cardinality $r(M(s))$ in G'. ■

Lemma 7 gives the following sufficient condition for $\Lambda_r(M(s)) = \Lambda_r([M(s)\ N(s)])$.

Lemma 8: Let G' be defined as in Lemma 7. Furthermore, let $\hat{V} = V \setminus \{v_i : i \in S\}$. If there exists a matching in the graph $\hat{G} = (W, \hat{V} \cup Y, E')$ with cardinality $|\hat{V} \cup Y|$, then $\Lambda_r(M(s)) = \Lambda_r([M(s)\ N(s)])$.

Proof: Suppose that there exists a matching $\hat{E}$ in $\hat{G}$ with cardinality $|\hat{V} \cup Y|$, so that all vertices in $\hat{V} \cup Y$ are matched. We can then select $E'' = \hat{E} \cup \{(u_i, v_i) : i \in S\}$, thus extending to a maximum matching on $G' = (W, V \cup Y, E')$. By Lemma 7, the minimum ξ_--weight of a maximum matching on $G(M(s))$ is zero, and hence the minimum ξ_--weight of a maximum matching on $G([M(s)\ N(s)])$ is zero as well, implying that $\Lambda_r(M(s)) = \Lambda_r([M(s)\ N(s)])$. ■

The condition of Lemma 8 can be expressed as a matroid constraint via the following theorem.

Theorem 3: Let $T = V \setminus S$. Define $\mathcal{M}_2 = (V, \mathcal{I})$ by $T \in \mathcal{I}$ if there is a matching of W into $V \cup Y$ in G' such that $T \cup Y$ is matched. Then $\mathcal{M}_2$ is a matroid, and $\Lambda_r(M(s)) = \Lambda_r([M(s)\ N(s)])$ if $T \in \mathcal{M}_1 \cap \mathcal{M}_2$.

Proof: The pair $\mathcal{M}_2$ defines a transversal matroid, as described in Section III-D. The set T is equal to the set $\hat{V}$ in Lemma 8, implying that $T \in \mathcal{M}_2$ iff there is a perfect matching in the graph $\hat{G} = (W, \hat{V} \cup Y, E')$. Hence if $T \in \mathcal{M}_1 \cap \mathcal{M}_2$, then $\Lambda_r(M(s)) = \Lambda_r([M(s)\ N(s)])$. ■

Remark 1: The condition of Lemma 8 also implies that there is a maximum matching from W into $V \setminus S$. This condition is equivalent to the dilation-free property required for structural controllability [1]; hence, if the graph induced by matrix A is connected, then structural controllability is also implied by Theorem 3.

C. Combined Problem Formulation

The sufficient conditions derived in Sections IV-A and IV-B lead to the following problem formulation. Let $\mathcal{M}_1$ be a transversal matroid with $T \in \mathcal{M}_1$ if there is a perfect matching from $W \setminus D$ into T, as in Theorem 2. Let $\mathcal{M}_2$ be a transversal matroid on the graph $G' = (W, V, E')$, where $E' = E \setminus \{(w_i, v_i) : A_{ii} = 0\}$, as in Theorem 3. Then a minimum-size input set that satisfies the sufficient conditions for disturbance decoupling and pole placement can be obtained by solving

$$\begin{array}{ll}\text{maximize} & |T| \\ T \subseteq V & \\ \text{s.t.} & T \in \mathcal{M}_1 \cap \mathcal{M}_2\end{array} \qquad (4)$$

and setting $S = V \setminus T$.

An alternative version of (4) can be obtained by assigning a weight $c(i)$ to each node i, representing the cost of controlling each state:

$$\begin{array}{ll}\text{maximize} & \sum_{i \in V} c(i) - \sum_{i \in T} c(i) \\ T \subseteq V & \\ \text{s.t.} & T \in \mathcal{M}_1 \cap \mathcal{M}_2\end{array} \qquad (5)$$

Again obtaining the input set via $S = V \setminus T$. Eqs. (4) and (5) define maximum-cardinality and maximum weight matroid intersection problems, and hence can be solved in polynomial time [5]. Efficient algorithms for both problems are presented in the following section.

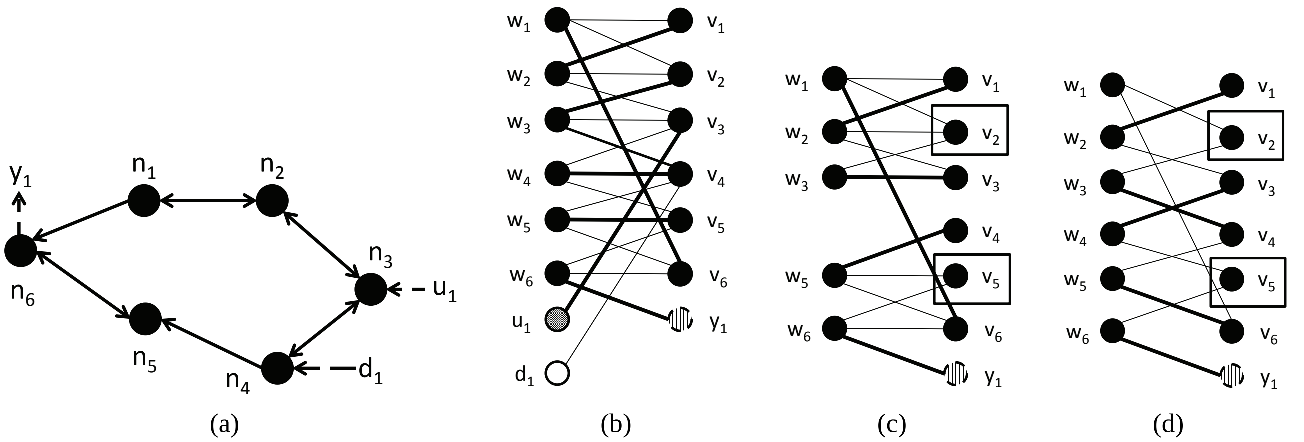


Fig. 1. Example of input selection for DDPP. (a) Network of $n = 6$ nodes with a single input, output, and disturbance. (b) Bipartite representation, with a maximum ξ_+-weight maximum matching that does not include the disturbance shown. The maximum ξ_+-weight maximum matching of $G(M(s))$ has weight 2, while the maximum ξ_+-weight maximum matching of $G([M(s)\ N(s)])$ has weight 3, implying that the DDPP conditions are not satisfied and new inputs must be selected. (c) Input selection to satisfy the condition $T \in \mathcal{M}_1$. A maximum matching of $W \setminus D$ into $V \cup Y$ is shown using bold lines. Setting $S = \{n_2, n_5\}$ implies that $V \setminus S = T \in \mathcal{M}_1$. (d) Input selection to satisfy the condition $T \in \mathcal{M}_2$. When $S = \{n_2, n_5\}$, there exists a maximum matching in G' (shown using bold lines), i.e., a maximum matching in which w_i is not matched to v_i for any i, thus implying that $T \in \mathcal{M}_2$.

V. Input Selection Algorithms for Disturbance Rejection

In this section, we present algorithms for solving the problems (4) and (5). Each problem can be solved efficiently using known, polynomial-time techniques for matroid intersection [5].

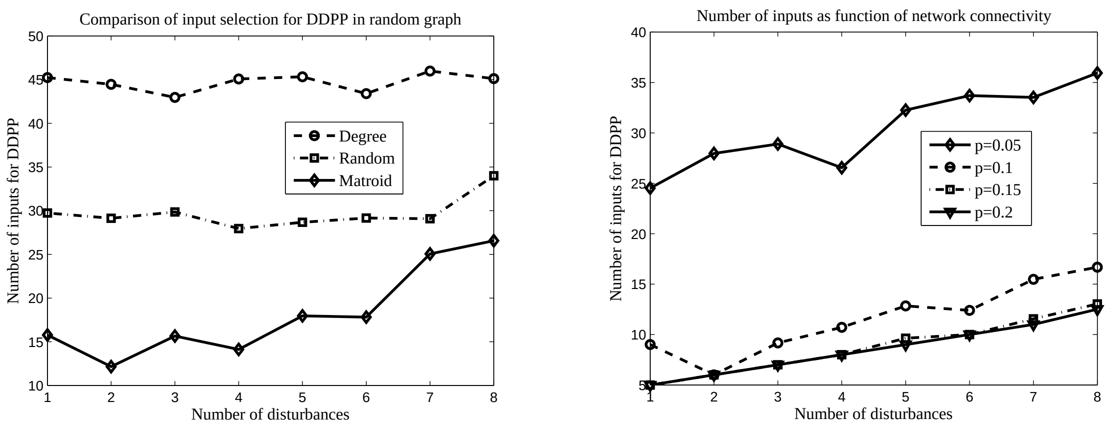
The approach of the algorithm for solving (4) is to attempt to increase the cardinality of T, with $T \in \mathcal{M}_1 \cap \mathcal{M}_2$, at each iteration, thus increasing the number of non-input nodes and decreasing the number of input nodes. In particular, at each iteration, the algorithm generates a bipartite graph based on the current value of T and attempts to find a shortest augmenting path in the graph. If such an augmenting path exists, then the augmenting path gives sets of nodes A_1 and A_2 such that $T \cup A_1 \setminus A_2$ increases the cardinality of T by one while remaining independent in both $\mathcal{M}_1$ and $\mathcal{M}_2$. A pseudocode description of this approach is given as Algorithm 1.

A key step in the algorithm is checking whether $T \in \mathcal{M}_1$ and $T \in \mathcal{M}_2$. The condition $T \in \mathcal{M}_1$ can be checked by determining whether there is a perfect matching of $W \setminus D$ into $T \cup Y$ in the graph G; such a matching can be found in polynomial time if it exists. Similarly, the condition $T \in \mathcal{M}_2$ can be checked by determining whether there is a perfect matching of W into $T \cup Y$ in the graph G'.

The runtime of the algorithm is known to be $O(n^{5/2})$ evaluations of the matroid constraints $T \in \mathcal{M}_1$ and $T \in \mathcal{M}_2$ [5]. Evaluating these constraints requires computing a maximum matching in a bipartite graph, which has complexity $O(n^{5/2})$, for total complexity $O(n^5)$ in the worst case. In order to solve (5), a modified version of Algorithm 1 can be employed. In the modification, the minimum-weight path P is found, instead of the shortest path.

Algorithm 1 Algorithm for the minimum-size input set for DDPP.

procedure DDPP(A, H, Q)
  Input: State matrix A, output matrix H, disturbance matrix Q
  Output: Set of input nodes S
  $T \leftarrow \emptyset$
  while 1 **do**
    $E(T) \leftarrow \emptyset$
    for $i \in T$, $j \notin T$ **do**
      if $(T - \{i\} \cup \{j\}) \in \mathcal{M}_1$ **then**
        $E(T) \leftarrow E(T) \cup \{(i,j)\}$
      end if
      if $(T - \{i\} \cup \{j\}) \in \mathcal{M}_2$ **then**
        $E(T) \leftarrow E(T) \cup \{(j,i)\}$
      end if
    end for
    $D(T) \leftarrow$ directed graph with vertices V and edge set $E(T)$
    $Z_1 \leftarrow \{j \in V \setminus T : (T \cup \{j\}) \in \mathcal{M}_1\}$
    $Z_2 \leftarrow \{j \in V \setminus T : (T \cup \{j\}) \in \mathcal{M}_2\}$
    if there is a path from Z_1 to Z_2 **then**
      $P \leftarrow$ shortest path from Z_1 to Z_2
      $T \leftarrow T \Delta P$
    else
      break
    end if
  end while
  $S \leftarrow V \setminus T$
  return S
end procedure



(a) (b)

Fig. 2. Numerical study of input selection for DDPP in a random graph with $n = 50$ and two randomly-selected output nodes. (a) Comparison of random, degree-based, and matroid intersection algorithms with $p = 0.075$. The matroid intersection algorithm required fewer input nodes than random input selection, which in turn required fewer than the degree-based selection. The minimum number of inputs increased linearly in the number of disturbances. (b) Impact of connectivity, determined by the probability of two nodes sharing a link p, on the number of inputs required for DDPP. Decreasing p from 0.2 to 0.15 does not increase the number of inputs required, but more inputs are needed as p decreases below 0.1.

VI. Input Selection Example

An example of input selection for DDPP is as follows. We consider a network of $n = 6$ nodes with a single output and disturbance, shown in Figure 1(a). We first evaluate the DDPP conditions when the node n_3 has an input, using the bipartite representation shown in Figure 1(b). We find that, in both $G(M(s))$ and $G([M(s)\ N(s)])$, the maximum cardinality of a matching is 7, and hence condition (i) is satisfied. On the other hand, the maximum ξ_+-weight maximum matching on $G(M(s))$ (shown as bold lines in Figure 1(b)) has weight 2, while the matching

$$E' = \{(d_1, v_4), (w_4, v_5), (w_5, v_6), (w_6, y_1), \\ (w_1, v_1), (w_2, v_2), (w_3, v_3)\}$$

is a maximum matching on $G([M(s)\ N(s)])$ with ξ_+-weight 3. Hence condition (ii) is not satisfied, and the DDPP property does not hold.

We now turn to selection of input nodes S to satisfy the conditions $V \setminus S = T \in \mathcal{M}_1 \cap \mathcal{M}_2$. We first examine the condition $T \in \mathcal{M}_1$ by constructing the bipartite graph with w_4 removed, shown as Figure 1(c). The matching $\{(w_1, v_6), (w_2, v_1), (w_3, v_3), (w_5, v_4), (w_6, y_1)\}$ is maximum and leaves n_2 and n_5 unmatched. We therefore select $S = \{n_2, n_5\}$ as a candidate input set, and have that $(V \setminus S) \in \mathcal{M}_1$.

Now, in order to verify that $T \in \mathcal{M}_2$, we consider the graph G' with all edges $\{(w_i, v_i) : i = 1, \ldots, 6\}$ removed (Figure 1(d)). The matching $\{(w_2, v_1), (w_3, v_4), (w_4, v_3), (w_5, v_6), (w_6, y_1)\}$ is a maximum matching onto $T \cup Y$ in G', and hence $T \in \mathcal{M}_2$. Since $T \in \mathcal{M}_1 \cup \mathcal{M}_2$, the input set $S = \{n_2, n_5\}$ satisfies the sufficient conditions for DDPP.

VII. Numerical Study

We evaluated our approach via numerical study using Matlab. For our study, we considered a directed random graph of $n = 50$ nodes, where each pair of nodes shared an edge independently with probability $p = 0.075$. The number of outputs was equal to 4. The number of nodes impacted by disturbances varied from 1 to 8. We compared the input selection algorithms of Section V with selecting random input nodes, as well as selecting high degree nodes as inputs. We also implemented a greedy algorithm, which approximates the solution to (4) by initializing T to be empty and, at each iteration, adding a node i to T satisfying $(T \cup \{i\}) \in \mathcal{M}_1 \cap \mathcal{M}_2$, provided such a node exists. For the algorithm of Section V, we included the additional restriction that nodes in D, as well as output nodes, cannot be chosen as inputs. Each data point represents an average over 15 independent trials.

Figure 2(a) shows a comparison of the minimum-size input set for different algorithms as the number of disturbances increases. The matroid intersection algorithm of Section V requires fewer input nodes than the other heuristics, although in some cases the greedy algorithm selects the same set of input nodes. The random selection algorithm selected fewer inputs than the degree-based heuristic, which had to select all nodes as inputs in order to achieve disturbance decoupling. This suggests that direct control of low-input nodes may be necessary to ensure DDPP.

The impact of the network connectivity on the number of input nodes is shown in Figure 2(b). The probability of two nodes sharing an edge increased from $p = 0.05$ to $p = 0.2$, while the number of disturbance nodes varied from 1 to 8. For larger values of p, resulting in more a more connected network, the number of inputs required is fewer than 10 and does not change significantly as p is decreased to 0.15. As p decreases from 0.1 to 0.05, the number of inputs for disturbance decoupling increases by a factor of 3, reflecting the fact that mitigating disturbances is substantially more difficult in a disconnected network.

VIII. Conclusions and Future Work

Networked cyber-physical systems operate in the presence of disturbances, including random noise, physical disturbances, and adversarial attacks. Robustness and stability under these disturbances is a basic requirement of CPS. When the CPS is controlled by a subset of input nodes, then the ability of the system to recover from disturbances is determined by the choice of input nodes.

In this paper, we considered the problem of selecting a subset of input nodes to guarantee the disturbance decoupling and pole placement (DDPP) property. DDPP implies that any disturbance to the system output can be canceled out by controlling the states of the input nodes, and that a controller can be designed to place the poles of the closed-loop system at arbitrary locations. Based on matching-based conditions for DDPP identified in [4], we derived two sufficient conditions for DDPP. We proved that each sufficient condition can be expressed as a matroid constraint on the set of non-input nodes, and also showed that one of the sufficient conditions implies the widely-studied matching-based characterization of structural controllability.

The problem of selecting a minimum-size set of input nodes for DDPP was then formulated as computing a maximum-cardinality intersection of two matroids, leading to efficient, polynomial-time algorithms for selecting the minimum-size set. An extension to selecting a minimum-cost set of input nodes, when different costs are introduced for choosing each node as an input, was formulated and a polynomial-time algorithm given. A numerical study suggested that the matroid intersection approach requires fewer input nodes to guarantee DDPP than random and degree-based algorithms, although a greedy heuristic also selected the minimum-size input set.

We note that the sufficient conditions for DDPP developed in this work are sufficient but not necessary. Developing tighter conditions, while still guaranteeing polynomial-time approximations, is one direction for future work, as is characterizing the optimality gap of our proposed approach. In addition, more general system models, including cases where the same disturbance affects multiple nodes, and systems with nonlinear dynamics, will be considered as generalizations of our approach. For systems where a large number of inputs are required to ensure disturbance decoupling, we will investigate weaker conditions that still provide some guarantees on disturbance rejection.

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