Towards Synchronization in Networks with Nonlinear Dynamics: A Submodular Optimization Framework

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***Abstract*—Synchronization underlies phenomena including memory and perception in the brain, coordinated motion of animal flocks, and stability of the power grid. These synchronization phenomena are often modeled through networks of phase-coupled oscillating nodes. Heterogeneity in the node dynamics, however, may prevent such networks from achieving the required level of synchronization. In order to guarantee synchronization, external inputs can be used to pin a subset of nodes to a reference frequency, while the remaining nodes are steered toward synchronization via local coupling. In this paper, we present a submodular optimization framework for selecting a set of nodes to act as external inputs in order to achieve synchronization from a desired set of initial states. We derive threshold-based sufficient conditions for synchronization, and then prove that these conditions are equivalent to constraints on monotone submodular functions over partition matroids. Based on this connection, we map the sufficient conditions for synchronization to constraints on submodular functions, leading to efficient algorithms with provable optimality bounds for selecting input nodes. We illustrate our approach via numerical studies of synchronization in power systems.**

I. Introduction

Synchronization plays a vital role in complex networks. Stable operation of the power grid requires synchronization of buses and generators to a common frequency [1]. Synchronized oscillations of neuronal firing provide a biological mechanism for aggregating information in perception [2] and memory [3]. Coordinated motion of animals [4] occurs when a common heading is achieved. The prevalence of synchronization across different application domains motivates the study of the basic principles underlying synchronization [5].

A variety of models have been proposed to quantify and characterize synchronization phenomena. These models include phase-coupled oscillator models such as the Kuramoto model [6], pulse-coupled synchronization models [7], consensus algorithms in distributed systems [8], and coupling models from statistical physics [9]. A common structure present in many of these models consists of an intrinsic dynamics of each node, together with a (nonlinear) diffusive coupling with the neighboring nodes. The relative strength of the intrinsic and diffusive coupling terms will determine whether the distributed nodes reach a synchronized or consensus state.

The existence and stability of synchronized states has been studied extensively in the literature [10], [11], [12], including conditions based on the intrinsic frequencies, network topology, and degree of coupling between the nodes. An important case is synchronization in the presence of external inputs [13]. External inputs arise in applications including neuroscience, where they represent environmental stimuli [2] or deep brain stimulation [14]. From an engineering standpoint, by introducing external inputs that pin a subset of nodes to a desired phase and frequency, a network that does not synchronize in the absence of inputs can be driven to a synchronized state, thus facilitating stability and performance of the network [15], [16].

Existing analytical approaches to introducing external inputs assume that the external input node is connected to all other nodes [15], or that the network has a specific topology such as a complete graph [16]. Developing sufficient conditions for synchronization using external inputs in networks with arbitrary topology is an open problem. Efficient algorithms for selecting a subset of input nodes in order to guarantee synchronization are also not available in the existing literature.

In this paper, we present an optimization framework for selecting a subset of nodes to act as external inputs in order to guarantee synchronization. Specifically, we consider networks where the dynamics of each node consists of a constant term and a nonlinear coupling with each neighboring node, and the input nodes are pinned to a fixed state. We develop a submodular optimization framework for selecting a minimum-size set of input nodes to achieve a desired level of synchronization, with provable bounds on the optimality of the chosen input nodes. Submodularity is a diminishing returns property of discrete optimization problems, analogous to convexity, that leads to provable optimality bounds for combinatorial problems. Our contributions are summarized as follows:

- We formulate the problem of selecting a minimum-size set of input nodes in order to ensure that the network converges to a fixed point within a desired bound of a reference state. We incorporate constraints that ensure that the node states remain within a desired region prior to convergence.

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- We derive a set of sufficient conditions for a given set of input nodes to achieve synchronization from any given set of initial states. We interpret our conditions as each node achieving a desired level of synchronization if a threshold number of neighbors reaches that level of synchronization.
- We develop a submodular optimization framework for selecting a minimum-size set of input nodes to achieve these synchronization conditions. We prove that each condition is equivalent to maximizing a monotone submodular function subject to a partition matroid constraint. We propose polynomial-time algorithms that exploit these structures in order to achieve provable bounds on the optimality of the chosen inputs. We interpret these bounds for the special case of Kuramoto dynamics.
- We evaluate our approach through a numerical study of synchronization in power grids using the IEEE 14 Bus test case [17]. We analyze the set of input nodes required for synchronization and the trajectories of the non-input nodes prior to synchronization.

The rest of the paper is organized as follows. In Section II, we give an overview of the related work. Section III contains our system model, definition of synchronization, and background on submodularity. We present our submodular framework for ensuring synchronization in Section IV. Input selection algorithms are described in Section V. Section VI contains our simulation results. Section VII concludes the paper and gives directions for future work.

II. Related Work

The phase-coupled oscillator framework for modeling synchronization phenomena was introduced in the seminal work of Winfree [18]. Models of oscillation include Lorenz [9], kick [19], Van der Pol [20], and pulse-coupled [7] oscillators. The sinusoidally-coupled Kuramoto model was introduced in [21]. Extensive studies have been performed on the mean-field behavior of the Kuramoto model with all-to-all coupling (i.e., each node is coupled to each other node) in the limit as the network size grows large [6]. Results on synchronization in the absence of any inputs can be found in [11], [22], [23], [23], [1].

Synchronization in the presence of external inputs has achieved relatively less study. Numerical studies have estimated the region of attraction, defined as the set of initial states that converge to the desired state, of the Kuramoto model with inputs for the case of all-to-all coupling [24]. Sufficient conditions for synchronization when there is a single input node that is connected to all other nodes were presented in [15], [25], [16]. These works do not, however, consider synchronization with external inputs in networks with arbitrary topology, and do not propose methods for selecting a subset of input nodes.

Steering a complex network to a desired state by pinning a set of nodes to a fixed value has been studied in the area of pinning control [26], [27], [28], [29]. These existing works assume that each node's dynamics is a nonlinear function of the current state with linear coupling between the neighbors, and hence differs from our approach that considers nonlinear coupling between neighboring nodes. To the best of our knowledge, conditions for pinning control of the model that we consider are not available in the existing literature.

Selecting external input nodes to achieve synchronization can be viewed as part of the broader area of selecting input nodes to control complex networks. Much recent work on selecting input nodes has focused on guaranteeing controllability of linear dynamics on the network [30], [31], [32], [33]. The assumption of linear dynamics, however, is not applicable to the nonlinear oscillators considered here. Furthermore, controllability analysis assumes the input node states can be set to any desired value, whereas our model considers input nodes that are pinned to a fixed reference state.

Submodularity has been used in the existing literature to solve graph optimization problems including influence maximization [34] and leader selection [35]. At present, however, there have been no applications of submodularity to nonlinear synchronization to the best of our knowledge.

III. Model and Preliminaries

In this section, we describe the system model and node dynamics, define the notion of synchronization considered in this paper, and give background on submodularity and matroids.

A. System Model

A network of n nodes, indexed in the set $V = \{1, \ldots, n\}$ is considered. Each node $v \in V$ has a neighbor set $N(v) \subseteq V$, consisting of the set of nodes that are coupled to v. We assume that links are bidirectional, so that $u \in N(v)$ implies $v \in N(u)$. An edge (u, v) exists if $u \in N(v)$ and $v \in N(u)$. We let E denote the set of edges. The graph $G = (V, E)$ is assumed to be connected; if not, our proposed input selection methods can be applied to each connected component of the graph.

Each node v has a time-varying state $x_v(t)$. The vector of node states at time t is denoted $\boldsymbol{x}(t) \in \mathbb{R}^n$. We assume that there are two types of nodes, denoted *input* and *non-input* nodes. We let A denote the set of input nodes. The states of the non-input nodes follow the nonlinear dynamics

$$\dot{x}_v(t) = -\sum_{u \in N(v)} \alpha_{uv}(x_v(t) - x_u(t)) + \omega_v. \tag{1}$$

In (1), the first term represents the coupling between the nodes, while ω_v describes the state dynamics in the absence of coupling. Note that the ω_v's may be distinct, corresponding to a case where the frequencies of the nodes are heterogeneous. The function $\alpha_{uv}(y)$ is assumed to satisfy $\frac{d\alpha_{uv}}{dy}(x_u - x_v) \geq 0$ when $\mathbf{x} \in \Lambda_{final}$. We define $\alpha'_{uv}(y) = \frac{d\alpha_{uv}}{dy}$.

As a motivating example, the Kuramoto model [6] can be obtained by setting $\alpha_{uv}(x_v - x_u) = K_{uv} \sin(x_v - x_u)$ for some coupling coefficient $K_{uv} \geq 0$. The condition $\frac{d\alpha_{uv}}{dy} \geq 0$ holds when $|x_v - x_u| \leq \frac{\pi}{2}$.

Each input node $v \in A$ is assumed to be pinned to a desired frequency ω_0 and state offset x_0, so that $\dot{x}_v(t) = \omega_0$ and $x_v(t) = \omega_0 t + x_0$ for all $v \in A$. We assume that $\omega_0 = x_0 = 0$.

This assumption is without loss of generality since we could define a new state $\hat{x}_v(t) = x_v(t) - \omega_0 t - x_0$ and $\hat{\omega}_i = \omega_i - \omega_0$. The overall node dynamics are given by

$$\dot{x}_v(t) = \left\{ \begin{array}{ll} -\sum_{u \in N(v)} \alpha_{uv}(x_v(t) - x_u(t)) + \omega_v, & v \notin A \\ \omega_0, & v \in A \end{array} \right. \quad (2)$$

We define Λ_{init} to denote the possible initial states, and Λ_{bound} to denote the set of safe states for the dynamics (2). The set Λ_{bound} is motivated by applications such as phase synchronization in power systems, where $|x_v - x_u| > \pi/2$ for two neighboring generators v and u implies that one generator will trip off.

We let $\overline{x}_v = \max\{|x_v| : \mathbf{x} \in \Lambda_{init}\}$ and x^* to be the largest value of r such that $||\mathbf{x}||_\infty \leq r$ implies that $\mathbf{x} \in \Lambda_{bound}$.

B. *Definition of Synchronization*

We now define the notion of synchronization considered in this paper.

Definition 1: A set of notes achieves γ-synchronization if, for any $\mathbf{x}(0) \in \Lambda_{init}$, (i) $\lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{x}^*$ with $\mathbf{x}^*$ satisfying $||\mathbf{x}^*||_\infty \leq \gamma$, and (ii) $\mathbf{x}(t) \in \Lambda_{bound}$ for each time t.
We define the set $\Lambda_{final} = \{\mathbf{x} : ||\mathbf{x}||_\infty \leq \gamma\}$.

This definition of synchronization implies that the node states converge to a fixed point with all node states within the desired bound γ of the reference state, 0. In the special case of the Kuramoto model, condition (i) is analogous to the practical synchronization condition of [1]. Condition (ii) ensures that the nodes never reach an undesired state, for example, having a phase separation of greater than $\pi/2$ in a power system.

C. *Background on Submodularity*

We now give background on submodularity and matroids. Let V denote a finite set, and let 2^V denote the set of all subsets of V.

Definition 2: A function $f : 2^V \rightarrow \mathbb{R}$ is *submodular* if, for any sets S and T with $S \subseteq T \subseteq V$ and any $v \in V \setminus T$,

$$f(S \cup \{v\}) - f(S) \geq f(T \cup \{v\}) - f(T). \quad (3)$$

Submodularity of a function $f(S)$ implies that the incremental increase in the function from adding an element v to a set S is larger than the benefit from adding v to a larger set T, and hence can be viewed as a notion of diminishing returns. For a submodular function $f(S)$, with $S \subseteq \{1, \ldots, n\}$, the multilinear relaxation $F : [0,1]^n \rightarrow \mathbb{R}$ is defined by

$$F(y) = \sum_{S \subseteq \{1,\ldots,n\}} \left[f(S) \left(\prod_{i \in S} y_i \right) \left(\prod_{i \notin S} (1 - y_i) \right) \right].$$

The function $F(y)$ can be interpreted as the expected value of $f(R)$, where R is a random subset of V that is obtained by sampling each element $i \in V$ with probability y_i.

A function $f : 2^V \rightarrow \mathbb{R}$ is *monotone* if $S \subseteq T$ implies that $f(S) \leq f(T)$.

A matroid is defined as follows.

Definition 3: A matroid $\mathcal{M} = (V, \mathcal{I})$ consists of a set V and a collection of subsets of V denoted $\mathcal{I}$. $\mathcal{M}$ is a matroid if (i) $\emptyset \in \mathcal{I}$, (ii) $A \subseteq B$ and $B \in \mathcal{I}$ implies $A \in \mathcal{I}$, and (iii) A, $B \in \mathcal{I}$ and $|A| < |B|$ implies that there exists $v \in B \setminus A$ with $(A \cup \{v\}) \in \mathcal{I}$.

If $\mathcal{M} = (V, \mathcal{I})$ is a matroid and $A \in \mathcal{I}$, then A is denoted an independent set. The rank of a matroid is defined as the size of the largest independent set. A maximal independent set is a *basis*. The matroid rank function $\rho : 2^V \rightarrow \mathbb{R}$ is defined as

$$\rho(A) = \max\{|B| : B \subseteq A \text{ and } B \in \mathcal{I}\},$$

i.e., the size of the largest independent subset of A. For any matroid $\mathcal{M}$ with $V = \{1, \ldots, n\}$, the matroid polytope $P(\mathcal{M}) \subseteq \mathbb{R}^n$ is defined by

$$\left\{ \mathbf{y} \in \mathbb{R}^n : \sum_{i \in A} y_i \leq \rho(A) \ \forall A \right\}.$$

A simple example of a matroid is the free matroid, defined by $S \in \mathcal{I}$ iff $|S| \leq k$ for some integer k. Another type of matroid that will be used in the subsequent sections is the partition matroid, defined as follows.

Definition 4: Let $V = V_1 \cup \cdots V_m$ be a partition of a set V, i.e., $V_i \cap V_j = \emptyset$ when $i \notin j$. Define a collection $\mathcal{I}$ by $A \in \mathcal{I}$ if $|A \cap V_i| \leq 1$ for all $i = 1, \ldots, m$. Then $\mathcal{M} = (V, \mathcal{I})$ is a partition matroid.

Lastly, we define the direct sum of two matroids.

Definition 5: Let $\mathcal{M}_1 = (V_1, \mathcal{I}_1)$ and $\mathcal{M}_2 = (V_2, \mathcal{I}_2)$. The direct sum $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ is defined by $V = V_1 \cup V_2$ and $A \in \mathcal{I}$ if $(A \cap V_1) \in \mathcal{I}_1$ and $(A \cap V_2) \in \mathcal{I}_2$.

IV. Submodular Framework for Synchronization

In this section, we formulate the problem of selecting a set of input nodes to guarantee synchronization. We first show that the nodes achieve synchronization if there exists a positive invariant set containing the set of possible initial states of the nodes, and if any initial state in the invariant set eventually converges to the set of desired final states, Λ_{final}. We then formulate a sufficient condition for existence of a positive invariant set and prove that it is equivalent to a submodular constraint. Finally, we formulate a sufficient condition for convergence to γ-synchronization, as well as an equivalent submodular constraint.

A. *Statement of Sufficient Condition*

We first present a known result on consensus in time-varying networks as a preliminary. For any $\delta > 0$ and any matrix B, the δ-digraph is defined as the digraph where edge (i, j) exists if $B_{ij} \geq \delta$.

Theorem 1 ([36], Theorem 1): Consider the linear system $\dot{x}(t) = F(t)x(t)$, where $F(t)$ is a time-varying system matrix. Assume that the system matrix is a bounded piecewise continuous function of time, and that for every time t the system matrix is Metzler (i.e., all off-diagonal elements are nonnegative) and has zero row sums. Suppose further that there is an index $k \in \{1, \ldots, n\}$, a threshold value $\delta > 0$, and an interval length $T > 0$ such that for all $t \in \mathbb{R}$ the δ-digraph associated to

$$\int_t^{t+T} F(s) \ ds$$

has the property that all nodes may be reached from the node k. Then the set of states $\{x^* \mathbf{1} : x^* \in \mathbb{R}\}$ is uniformly exponentially stable. In particular, all components of any solution converge to a common value as $t \rightarrow \infty$.

The following theorem gives a sufficient condition for a set of input nodes to guarantee synchronization.

Theorem 2: Suppose that there exists a set Λ_{PI} such that the following conditions hold: (a) $\Lambda_{init} \subseteq \Lambda_{PI} \subseteq \Lambda_{bound}$, (b) Λ_{PI} is *positive invariant*, i.e., if $\boldsymbol{x}(0) \in \Lambda_{PI}$, then $\boldsymbol{x}(t) \in \Lambda_{PI}$ for all $t \geq 0$, and (c) If $\boldsymbol{x}(0) \in \Lambda_{PI}$, then there exists T such that $t \geq T$ implies $\boldsymbol{x}(t) \in \Lambda_{final}$. Then the set of input nodes A satisfies conditions (i) and (ii) of Definition 1 and hence ensures γ-synchronization.

Proof: In order to prove condition (i) of Definition 1, consider $\dot{\mathbf{x}}(t)$, which has dynamics

$$\ddot{x}_v(t) = -\sum_{u \in N(v)} [\alpha'_{uv}(x_v(t) - x_u(t))(\dot{x}_v(t) - \dot{x}_u(t))] \quad (4)$$

for $v \notin A$ and $\ddot{x}_v(t) \equiv 0$ for $v \in A$.

We now define dynamics of the form in Theorem 1 in order to analyze the convergence of (4). Let $z(t) \in \mathbb{R}^{n+1}$ denote the state variable, where z_{n+1} is the state of a "super node" with dynamics $\dot{z}_{n+1}(t) \equiv 0$. Suppose that $t > T$ and hence $\mathbf{x}(t) \in \Lambda_{final}$. Define the system matrix $F(t)$ by

$$F_{vu}(t) = \begin{cases} \alpha'_{uv}(x_v(t) - x_u(t)), \text{ for } (u,v) \in E, v \notin A \\ -\sum_{s \in N(v)} \alpha'_{sv}(x_v(t) - x_s(t)), \ u = v, u \notin A \\ -1, \text{ for } u = v, u \in A \\ 1, \text{ for } v \in A, u = (n+1) \\ 0, \text{ for } v = n+1 \end{cases}$$

By assumption, $\alpha'_{uv} \geq 0$. Hence $F(t)$ is a bounded, piecewise continuous Metzler matrix with rows that sum to zero, and the connectivity of the graph G implies that node $(n+1)$ is connected to all other nodes in the associated δ-digraph. By Theorem 1, $z(t)$ converges to a state $x^*\mathbf{1}$.

If we set $z_v(0) = z_{n+1}(0) = 0$ for all $v \in A$ and $z_v(0) = \dot{x}_v(0)$ for all $v \notin A$, then the trajectory of $[z_1(t) \cdots z_n(t)]^T$ will be identical to the trajectory of $\dot{\boldsymbol{x}}(t)$. Hence, by Theorem 1, $\lim_{t \rightarrow \infty} \dot{\boldsymbol{x}}(t) = \omega^* \mathbf{1}$ for some $\omega^* \in \mathbb{R}$. Moreover, since $\dot{x}_v(t) \equiv 0$ for all $v \in A$, we must have $\omega^* = 0$, implying convergence to a fixed point.

Since $\dot{\boldsymbol{x}}$ converges to 0, $\boldsymbol{x}(t)$ converges to a fixed point $\boldsymbol{x}^*$. By condition (c), since $\boldsymbol{x}(t) \in \Lambda_{final}$ for t sufficiently large, $\boldsymbol{x}^* \in \Lambda_{final}$. This completes the proof of condition (i) of Definition 1.

The proof of condition (ii) in Definition 1 follows from (a) and (b), which imply that $\boldsymbol{x}(t) \in \Lambda_{PI} \subseteq \Lambda_{bound}$ for all $t \geq 0$ whenever $\boldsymbol{x}(0) \in \Lambda_{init}$. ■

Figure 1 illustrates the relationship between the sets Λ_{bound}, Λ_{init}, Λ_{final}, and Λ_{PI} of Definition 1 and Theorem 2.

The following two subsections present a submodular formulation for selecting a set of input nodes to satisfying the conditions of Theorem 2.

B. Submodular Constraint for Existence of Positive Invariant Set

In this section, we formulate a submodular constraint for existence of a subset Λ_{PI} that is positive invariant and satisfies

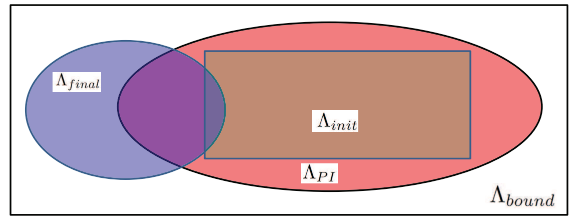


Fig. 1. Illustration of the sets Λ_{bound}, Λ_{init}, Λ_{final}, and Λ_{PI}.

$\Lambda_{init} \subseteq \Lambda_{PI} \subseteq \Lambda_{bound}$. This corresponds to conditions (a) and (b) of Theorem 2. We first state the following proposition that gives a sufficient condition for positive invariance.

Proposition 1: Suppose that a set Λ_{PI} is defined by $\Lambda_{PI} = \{\boldsymbol{x} : |x_v| \leq x_v^{PI}\}$ for some $\{x_v^{PI} : v \in V\}$. If

$$\sum_{u \in N(v)} \min\left\{\alpha_{uv}(x_v^{PI} - x_u) : |x_u| \leq x_u^{PI}\right\} > |\omega_v| + \epsilon \quad (5)$$

for all $v \in V$ and some $\epsilon > 0$, then the set Λ_{PI} is positive invariant.

Proof: Suppose that (5) holds for all $v \in V$ and yet Λ_{PI} is not positive invariant. Let $\boldsymbol{x}(0) \in \Lambda_{PI}$ be such that $\boldsymbol{x}(t) \notin \Lambda_{PI}$ for some t, and define

$$t^* \triangleq \inf\{t : x_v(t) > x_v^{PI} \text{ for some v}\}.$$

Then $|x_v(t^*)| = x_v^{PI}$ for some $v \in V$, and we have

$$\begin{aligned} \dot{x}_v(t^*) &= -\sum_{u \in N(v)} \alpha_{uv}(x_v^{PI} - x_u(t^*)) + \omega_v \\ &< -\sum_{u \in N(v)} \min\left\{\alpha_{uv}(x_v^{PI} - x_u) : \right. \\ &\qquad \left. |x_u| \leq x_u^{PI}\right\} + |\omega_v| \\ &< -(|\omega_v| + \epsilon) + |\omega_v| < -\epsilon \end{aligned}$$

implying that $|x_v(t)| \leq x_v^{PI}$ in some neighborhood of t^* and contradicting the definition of t^*.

A similar argument holds when $x_v(t^*) = -\overline{x}_v$. We have

$$\begin{aligned} \dot{x}_v(t^*) &= -\sum_{u \in N(v)} \alpha_{uv}(-x_v^{PI} - x_u(t^*)) + \omega_v \\ &= \sum_{u \in N(v)} \alpha_{uv}(x_v^{PI} + x_u(t^*)) + \omega_v \\ &> \sum_{u \in N(v)} \min\left\{\alpha_{uv}(x_v^{PI} - x_u) : \right. \\ &\qquad \left. |x_u| \leq x_u^{PI}\right\} + \omega_v \\ &> |\omega_v| + \epsilon - |\omega_v| > \epsilon \end{aligned}$$

which implies that $x_v(t) > -x_v^{PI}$ in some neighborhood of t^*. These contradictions imply Λ_{PI} is positive invariant. ■

Based on this sufficient condition, we can evaluate the number of input nodes required for synchronization for certain types of graphs, as in the following lemma.

Lemma 1: Suppose that the graph $G = (V, E)$ is complete, i.e., there exists an edge $(u, v) \in E$ for each u and v in V, and that $K_{uv} \equiv K$ for all u and v. Then synchronization is achieved if $x_v^{max} \leq \frac{\pi}{4}$ and $|A| \geq \frac{|\omega|\sqrt{2}}{K}$.

Proof: Choose as the candidate positive invariant set $\Lambda_{PI} = \left\{|x_v| \leq \frac{\pi}{4}\right\}$. Then Proposition 1 implies that synchronization is achieved if

$$\sum_{u \in A} \sin \frac{\pi}{4} \geq |\omega|,$$

which is equivalent to $|A| \geq \frac{|\omega|\sqrt{2}}{K}$. ■

As a first step in our approach, we develop a discrete version of the constraint in Proposition 1. In what follows, we assume that $\alpha_{uv}'(x) \geq 0$ for all x; the general case is considered in Section IV-D. Let M be a positive integer. For any integers $m_u, m_v \in \{1, \ldots, M\}$, define

$$\lambda_{uv}(m_u, m_v) = \min \left\{ \alpha_{uv}\left(\frac{m_v x_v^{max}}{M} - x_u\right) : |x_u| \leq \frac{m_u x_u^{max}}{M} \right\}.$$

By Proposition 1, a sufficient condition for existence of a positive invariant set is the existence of a set of integers $(m_v : v \in V)$ satisfying

$$\sum_{u \in N(v)} \lambda_{uv}(m_u, m_v) \geq |\omega_v| + \epsilon.$$

Note that the construction $x_v^{PI} = \frac{m_v x_v^{max}}{M}$ and $m_v \leq M$ ensures that $x_v^{PI} \leq x_v^{max}$, and hence $\Lambda_{PI} \subseteq \Lambda_{bound}$.

We can obtain an equivalent condition by setting

$$\rho_{uv}(m_u, m_v) = \lambda_{uv}(m_u, m_v) - \min_{m_u, m_v = 1, \ldots, M} \lambda_{uv}(m_u, m_v),$$

given by

$$\sum_{u \in N(v)} \rho_{uv}(m_u, m_v) \geq |\omega_v| + \epsilon - \sum_{u \in N(v)} \min_{m_u, m_v = 1, \ldots, M} \lambda_{uv}(m_u, m_v) \triangleq \tau_v, \quad (6)$$

noting that $\rho_{uv}(m_u, m_v) \geq 0$ for all m_u and m_v.

We now formulate a submodular constraint for ensuring convergence to a positive invariant set. Our approach is to introduce the set $\overline{V} = \{v_m : m = \overline{m}_v, \ldots, M\}$, where

$$\overline{m}_v = \min \{m : |x_v| \leq \frac{x_v^{max} m}{M} \ \forall \mathbf{x} \in \Lambda_{init}\},$$

and to map the problem of identifying a positive invariant set to selecting a set $S \subseteq \overline{V}$ subject to constraints that we will show are equivalent to (6). Under this mapping, $v_m \in S$ implies that $m_v = m$. The definition of $\overline{V}$ ensures that $m_v \geq \overline{m}_v$ for all $v \in V$, and hence that $\Lambda_{init} \subseteq \Lambda_{PI}$.

In order to ensure that the mapping is well-defined and that the number of inputs does not exceed a desired bound k, we introduce the collection $\mathcal{I}_k$ of subsets of $\overline{V}$, with $S \in \mathcal{I}_k$ if $|S \cap \{v_m : m = 1, \ldots, M\}| \leq 1$ for all $v \in V$ and $|S \cap \{v_0 : v \in V\}| \leq k$.

Lemma 2: The collection $\mathcal{M}_k = (\overline{V}, \mathcal{I}_k)$ is a matroid.

Proof: Define $\overline{V}^{(0)} = \{v_0 : v \in V\}$ and $\overline{V}^{(1)} = \{v_m : v \in V, m = \overline{m}_v, \ldots, M\}$. We construct a collection of subsets of $\overline{V}^{(0)}$, denoted $\mathcal{I}_k^{(0)}$, by $S \in \mathcal{I}_k^{(0)}$ if $|S| \leq k$. Similarly, construct a collection of subsets of $\overline{V}^1$, denoted $\mathcal{I}_k^{(1)}$, by $S \in \mathcal{I}_k^{(1)}$ if $|S \cap \{v_m : m = \overline{m}_v, \ldots, M\}| \leq 1$ for all $v \in V$.

The collection $\mathcal{M}_k^{(0)} = (\overline{V}^{(0)}, \mathcal{I}_k^{(0)})$ is a free matroid of cardinality k. The collection $\mathcal{M}_k^{(1)} = (\overline{V}^{(1)}, \mathcal{I}_k^{(1)})$ is a partition matroid. By inspection, $\mathcal{M}_k = \mathcal{M}_k^{(1)} \oplus \mathcal{M}_k^{(2)}$, and hence $\mathcal{M}_k$ is a matroid as well. ■

The set of bases of $\mathcal{M}_k$ is denoted $\mathcal{B}_k$. We observe that a set $S \in \mathcal{I}_k$ could have $v_0 \in S$ and $v_m \in S$. In this case, the quantity m_v from Eq. (6) is given by $m_v = 0$.

We will now construct a submodular function $f_v(S)$ such that $\{f_v(S) \geq \tau_v\} \cap \{S \in \mathcal{I}_k\}$ is equivalent to (6). As a preliminary, define

$$S_{uv} = S \cap \{u_l, v_m : l = \overline{m}_u, \ldots, M, m = \overline{m}_v, \ldots, M\}.$$

One candidate function is given by

$$\sum_{u_{m'}, v_m \in S} \rho_{uv}(u_{m'}, v_m).$$

This function, however, is not submodular as a function of S; instead, if $v_0 \notin S$, we define $f_v(S) = \sum_{u \in N(v)} f_{uv}(S)$, where

$$f_{uv}(S) = K_{uv} \log(|S_{uv}| + 1) + \max \left\{ \sum_{\substack{v_m \in S \\ u_{m'} \in S}} \rho_{uv}(u_{m'}, v_m), g_{uv}(S) \right\}. \quad (7)$$

In (7), $g_{uv}(S) = \min\{\rho_{uv}(0, v_m) : v_m \in S\}$ if $u_0 \in S$ and 0 if $u_0 \notin S$. The parameter K_{uv} is defined by

$$K_{uv} = \frac{1}{\log \phi} \sum_{u_m', v_m \in \overline{V}} \rho_{uv}(u_{m'}, v_m) \triangleq \frac{1}{\log \phi} Z_{uv} \quad (8)$$

for some $\phi \in \left(1, 1 + \frac{1}{4M^2 + 8M + 3}\right)$. If $v_0 \in S$, then let $f_v(S) = R$ for some R satisfying $R \geq f_v(S)$ for all S that do not contain v.

Let $K_v = \sum_{u \in N(v)} K_{uv}$. We now prove that Eq. (6) can be expressed as a constraint on $f_v(S)$.

Proposition 2: There exist a set of indices $(m_v : v \in V)$ and a set of inputs A with $|A| \leq k$ such that (6) holds for all $v \in V$ if and only if there is a set S satisfying $S \in \mathcal{B}_k$ and $f_v(S) \geq K_v \log 3 + \tau_v$ for all $v \in V$.

Proof: First, suppose that the indices $(m_v : v \in V)$ satisfy (6) with input set A. We assume without loss of generality that $|A| = k$ (otherwise we can add elements to A until $|A| = k$). For each $v \in A$ (for which $m_v = 0$), choose an arbitrary $\hat{m}_v \in \{\overline{m}_v, \ldots, M\}$. Define the set S by

$$S = \{v_{m_v} : v \in V \setminus A\} \cup \{v_0, v_{\hat{m}_v} : v \in A\}.$$

By construction, $S \in \mathcal{B}_k$. We have $|S_{uv}| = 2$ for all u and v.

The functions $f_{uv}(S)$ for $v \notin A$ are evaluated as follows. If $u \notin A$, then $f_{uv}(S) = K_{uv} \log 3 + \rho_{uv}(u_{m_u}, v_{m_v})$. If $u \in A$, then

$$\begin{aligned} f_{uv}(S) &= \max\{\rho_{uv}(u_{\hat{m}_u}, m_v), \rho_{uv}(0, m_v)\} \\ &= \rho_{uv}(0, m_v) = \rho_{uv}(m_u, m_v), \end{aligned}$$

since $\rho_{uv}(\hat{m}_u, m_v) \leq \rho_{uv}(0, m_v)$ by definition. Hence

$$\sum_{u \in N(v)} f_{uv}(S) = K_v \log 3 + \sum_{u \in N(v)} \rho_{uv}(m_u, m_v) \geq K_v \log 3 + \tau_v$$

for all $v \in V$.

Conversely, suppose that $S \in \mathcal{B}_k$ and $f_v(S) \geq d_v K \log 3 + \tau_v$ for all $v \in V$. For each $v \in V$, set $m_v = 0$ if $v_0 \in S$ and m_v as the unique integer with $v_{m_v} \in S$ otherwise. By the same argument as above,

$$f_{uv}(S) = K_{uv} \log 3 + \rho_{uv}(m_u, m_v),$$

and hence $f_v(S) \geq K_v \log 3 + \tau_v$ is equivalent to (6). ∎

We next prove that $f_v(S)$ is monotone and submodular as a function of S, starting with monotonicity.

Lemma 3: The function $f_v(S)$ is monotone nondecreasing as a function of S.

The proof is given in the appendix. The next step is to show that $f_{uv}(S)$ is submodular as a function of S.

Proposition 3: The function $f_{uv}(S)$ is submodular as a function of S.

The proof can be found in the appendix. The following lemma enables us to formulate existence of a positive invariant set as a single matroid constraint and a single submodular constraint.

Lemma 4: The constraint $f_v(S) \geq K_v \log 3 + \tau_v$ for all $v \in V$ is equivalent to

$$\tilde{f}(S) \triangleq \sum_{v \in V} \min\{f_v(S), \tau_v + K_v \log 3\} \geq \sum_{v \in V} (\tau_v + K_v \log 3).$$

The function $\tilde{f}(S)$ is monotone nondecreasing and submodular as a function of S.

Proof: If $f_v(S) \geq K_v \log 3 + \tau_v$ for all $v \in V$, then each term of $F(S)$ is equal to $\tau_v + K_v \log 3$ and hence $F(S) \geq \sum_{v \in V} (\tau_v + K_v \log 3)$. Conversely, $\tilde{f}(S)$ is bounded above by $\sum_{v \in V} (\tau_v + K_v \log 3)$, and equality can hold only if $f_v(S) \geq K_v \log 3 + \tau_v$ for each $v \in V$.

Since $f_v(S)$ is submodular as a function of S, it is a known result that $\min\{f_v(S), c\}$ is submodular as a function of S for any constant c. Hence $\tilde{f}(S)$ is a sum of submodular functions and is submodular. Monotonicity of $\tilde{f}(S)$ follows from monotonicity of each $f_v(S)$. ∎

Lemma 4 and Proposition 2 imply that (6) is equivalent to the existence of a set $S \subseteq \overline{V}$ satisfying $S \in \mathcal{I}_k$ and $F(S) \geq \sum_{v \in V} K_v \log 3 + \tau_v$.

C. Ensuring Convergence to Synchronization

The following proposition gives sufficient conditions to ensure that, for any $\boldsymbol{x}(0) \in \Lambda_{init}$, there exists T such that $\boldsymbol{x}(t) \in \Lambda_{final}$ for all $t \geq T$. This corresponds to condition (c) of Theorem

Proposition 4: Let $\Lambda = \{\boldsymbol{x} : |x_v| \leq \overline{x}_v\}$ be a positive invariant set. Suppose there exists an index v and $\underline{x}_v \in [0, \overline{x}_v]$ such that

$$\sum_{u \in N(v)} \min\{\alpha_{uv}(x_v - x_u) : x_v \in [\underline{x}_v, \overline{x}_v], |x_u| \leq \overline{x}_u\} > |\omega_v| + \epsilon. \quad (9)$$

Define $\Lambda' = \Lambda \setminus \{\boldsymbol{x} : x_v \in [\underline{x}_v, \overline{x}_v]\}$. Then Λ' is positive invariant and, when $\boldsymbol{x}(0) \in \Lambda$, there exists T such that $\boldsymbol{x}(t) \in \Lambda'$ for all $t \geq T$.

Proof: We first show positive invariance of Λ'. Suppose $\boldsymbol{x}(0) \in \Lambda'$. Since Λ is positive invariant, $|x_u| \leq \overline{x}_u$ for all $u \neq v$. It suffices to show that if $|x_v(0)| \leq \underline{x}_v$, then $|x_v(t)| \leq \underline{x}_v$ for all $t \geq 0$. This result holds by (9) and Proposition 1.

It remains to prove that, if $\boldsymbol{x}(0) \in \Lambda$, then there exists T such that $\boldsymbol{x}(t) \in \Lambda'$ for $t \geq T$. Suppose that this is not the case, and choose $\boldsymbol{x}(0) \in \Lambda$ such that $x_v(t) \in [\underline{x}_v, \overline{x}_v]$ for all $t \geq 0$. Then

$$\dot{x}_v(t) = -\sum_{u \in N(v)} \alpha_{uv}(x_v(t) - x_u(t)) + \omega_v \leq -|\omega_v| - \epsilon + |\omega_v| = -\epsilon,$$

implying that

$$x_v(t) \leq x_v(0) - \epsilon t \leq \overline{x}_v - \epsilon t$$

and hence $x_v(t) \leq \underline{x}_v$ for t sufficiently large. This contradiction implies that $\boldsymbol{x}(T) \in \Lambda'$ for some $T > 0$. Positive invariance of Λ' then yields $\boldsymbol{x}(t) \in \Lambda'$ for $t \geq T$. ∎

Based on Proposition 4, we divide the time into intervals of length T, and let

$$m_v^i = \max\left\{m : |x_v(t)| \leq \frac{m\gamma}{M} \ \forall t \leq iT\right\}.$$

Hence m_v^i defines on upper bound on $x_v(t)$ during the first i time intervals. A sufficient condition for (9) is then given by

$$\sum_{u \in N(v)} \sigma_{uv}(m_u^{i-1}, m_v^i) \geq |\omega_v| + \epsilon,$$

where

$$\sigma_{uv}(m, l) = \min\{\alpha_{uv}(x_v - x_u) : x_v \in \left[\frac{(m-1)x_v^{max}}{M}, \frac{m x_v^{max}}{M}\right], |x_u| \leq \frac{l x_u^{max}}{M}\right\}.$$

We observe that the values of m_v^i are nonincreasing over time, and that during each time epoch, either at least one value of m_v^i is decremented, or no values of m_v^j can be decremented in any future epoch $j > i$. Since each value of m_v^i can be decremented at most M times, the total number of epochs that must be considered is bounded by $(MN+1)$.

Convergence to Λ_{final} is then ensured if

$$m_v^{MN+1} \leq \max\left\{m : \frac{m x_v^{max}}{M} \leq \gamma\right\} \triangleq \underline{m}_v.$$

In what follows, we develop a submodular optimization approach for ensuring convergence to Λ_{final}. Analogous to Section IV-B, we define the ground set

$$\begin{aligned}\tilde{V} = &\{v_{m}^{0} : m = \overline{m}_{v}, \ldots, M, v \in V\} \\ &\cup \{v_{m}^{i} : m = 1, \ldots, M, v \in V, i = 1, \ldots, MN\} \\ &\cup \{v_{m}^{MN+1} : m = 1, \ldots, \underline{m}_{v}, v \in V\} \cup \{v_{0} : v \in V\}.\end{aligned}$$

For a set $S \subseteq \overline{V}$, $v_{m}^{i} \in S$ implies that $m_{v}^{i} \leq m$. The constraint that $v_{m}^{0} \in \tilde{V}$ only for $m \geq \overline{m}_{v}$ ensures that all initial states in Λ_{init} are included in the analysis. Similarly, $v_{m}^{MN+1} \in \tilde{V}$ for $m \leq \underline{m}_{v}$ ensures that the final states lie in Λ_{final}.

We define the set of feasible sets $\mathcal{J}_{k}$ by $S \in \mathcal{J}_{k}$ if

$$|S \cap \{v_{m}^{i} : m = 1, \ldots, M\}| \leq 1$$

for all $v \in V$ and $i = 0, \ldots, MN+1$, and if $|S \cap \{v_{0} : v \in V\}| \leq k$.

Lemma 5: The collection of sets $\mathcal{N}_{k} = (\tilde{V}, \mathcal{J}_{k})$ is a matroid.

Proof: Define $\overline{\mathcal{N}}_{k}$ by $\overline{\mathcal{N}}_{k} = (\tilde{V}^{\prime}, \mathcal{J}_{k}^{\prime})$, where $\tilde{V}^{\prime} = \{v_{0} : v \in V\}$, and $S \in \mathcal{J}_{k}^{\prime}$ if $|S| \leq k$. Define $\mathcal{N}_{k}^{i}$ by $\mathcal{N}_{k}^{i} = (\tilde{V}_{k}^{i}, \mathcal{J}_{k}^{i})$, with $\tilde{V}_{k}^{i} = \{v_{m}^{i} : i = 1, \ldots, M, v \in V\}$ and $S \in \mathcal{J}_{k}^{i}$ if $|S \cap \{v_{m}^{i} : m = 1, \ldots, M\}| \leq 1$ for all $v \in V$.

By construction, $\overline{\mathcal{N}}_{k}$ is a free matroid of rank k, and each $\mathcal{N}_{k}^{i}$ is a partition matroid. Furthermore,

$$\mathcal{N}_{k} = \overline{\mathcal{N}}_{k} \oplus \mathcal{N}_{k}^{0} \oplus \cdots \oplus \mathcal{N}_{k}^{MN+1},$$

implying that $\mathcal{N}_{k}$ is a direct sum of matroids and hence is a matroid. ∎

We now construct a submodular function that gives a sufficient condition for convergence to Λ_{final}. For each $i = 1, \ldots, MN+1$ and $(u,v) \in E$, define

$$f_{uv}^{i}(S) = K_{uv} \log\left(|S_{uv}^{i}| + 1\right) + \max\left\{\sum_{\substack{v_{m}^{i} \in S \\ u_{m^{\prime}}^{i-1} \in S}} \sigma_{uv}(m^{\prime}, m), g_{uv}^{i}(S)\right\},$$

where K_{uv} is defined as

$$K_{uv} = \frac{1}{\log \phi} \sum_{m, m^{\prime} = 1, \ldots, M} \sigma_{uv}(m^{\prime}, m) \triangleq Z_{uv},$$

the set S_{uv} is defined by

$$S_{uv}^{i} = S \cap \left(\{u_{m}^{i-1} : m = 1, \ldots, M\} \cup \{v_{m}^{i} : m = 1, \ldots, M\}\right),$$

and

$$g_{uv}^{i}(S) = \left\{\begin{array}{ll} \min_{v_{m}^{i} \in S}\{\sigma uv(0, m)\}, & u_{0} \in S \\ 0, & \text{else} \end{array}\right.$$

The following proposition provides a sufficient condition for convergence to Λ_{final} based on $f_{v}^{i}(S)$.

Proposition 5: If there exists S such that $S \in \mathcal{B}(\mathcal{N}_{k})$ and

$$\begin{aligned} f_{v}^{i}(S) \geq\ & K_{v} \log 3 + |\omega_{v}| + \epsilon \\ & - \sum_{u \in N(v)} \min\{\alpha(x_{v} - x_{u}) : \mathbf{x} \in \Lambda_{bound}\} \\ \triangleq\ & \beta_{v} \end{aligned}$$

for all $v \in V$ and $i = 1, \ldots, MN+1$, then the state $\mathbf{x}(t)$ converges to Λ_{final}.

Proof: Define m_{v}^{i} to be equal to 0 if $v_{0} \in V$, and equal to the unique v with $v_{m}^{i} \in S$ otherwise. We will show that, at time Ti, $|x_{v}(t)| \leq \frac{m_{v}^{i} x_{v}^{max}}{M}$ by induction on i. At time $i = 0$, the condition holds by assumption. Suppose the result holds up to time $i-1$, and at time i we have $f_{v}^{i}(S) \geq \beta_{v}$ for all $v \in V$. Then

$$\begin{aligned} f_{v}^{i}(S) &= \sum_{u \in N(v)} (K_{uv} \log 3 + \sigma uv(m_{u}^{i-1}, m_{v}^{i})) \\ &\geq K_{v} \log 3 + |\omega_{v}| + \epsilon \\ &\quad - \sum_{u \in N(v)} \min\{\alpha(x_{v} - x_{u}) : \mathbf{x} \in \Lambda_{bound}\}. \end{aligned}$$

Equivalently,

$$\begin{aligned} &\sum_{u \in N(v)} \min\{\alpha_{uv}(x_{v} - x_{u}) : \\ &\quad x_{v} \in \left[\frac{(m_{v}^{i}) x_{v}^{max}}{M}, \frac{m_{v}^{i-1} x_{v}^{max}}{M}\right], |x_{u}| \leq \frac{m_{u}^{i-1} x_{u}^{max}}{M}\right\} \\ &\qquad > |\omega_{v}| + \epsilon, \end{aligned}$$

implying that $|x_{v}(t)| \leq \frac{m_{v}^{i} \gamma}{M}$ for $t \geq iT$ by Proposition 4. Hence, by induction, for t sufficiently large,

$$|x_{v}(t)| \leq \frac{m_{v}^{MN+1} x_{v}^{max}}{M} \leq \frac{\overline{m}_{v} x_{v}^{max}}{M} \leq \gamma$$

since $v_{m}^{MN+1} \in S$ and $S \in \mathcal{B}(\mathcal{N}_{k})$ implies that $v_{m} \leq \underline{m}_{v}$. ∎

The following result enables convergence to Λ_{final} to be formulated as a submodular constraint on S.

Proposition 6: The function $f_{v}^{i}(S)$ is monotone and submodular as a function of S.

The proof can be found in the appendix. In order to combine the results of this section and the previous section, we define a function

$$f(S) \triangleq \tilde{f}(S) + \sum_{i=1}^{MN+1} \sum_{v \in V} \min\{f_{v}^{i}(S), \beta_{v}\}, \quad (10)$$

which is monotone and submodular as a function of S since $\tilde{f}(S)$ and each $f_{v}^{i}(S)$ is monotone and submodular. The problem of selecting a minimum-size set of input nodes to ensure synchronization can then be formulated as

$$\begin{array}{ll} \text{minimize} & k \\ \text{s.t.} & f(S) \geq \sum_{v \in V} \tau_{v} + \sum_{i=1}^{MN+1} \sum_{v \in V} \beta_{v} \\ & S \in \mathcal{N}_{k} \end{array} \quad (11)$$

D. Input Selection with Non-Monotone Coupling

The case where $\frac{d\alpha_{uv}}{dx}$ can be negative is considered as follows. An example of a scenario where this occurs is the Kuramoto model when the phase separation between adjacent nodes exceeds $\pi/2$. As in [15], we develop conditions for synchronization in which the set of input nodes is a *dominating set*, i.e., $A \cap \{u, v\} \neq \emptyset$ for each $(u, v) \in E$. The following proposition gives sufficient conditions for synchronization when this assumption holds.

Proposition 7: If the functions α_{uv} satisfy $\frac{d\alpha_{uv}}{dx} \geq 0$ for $x \in \Lambda_{final}$, the non-input nodes v satisfy

$$\sum_{u \in N(v)} \alpha_{uv}(x_v) > |\omega_v|$$

for all $\mathbf{x} \in \Lambda_{init} \backslash \Lambda_{final}$, and $A \cap \{u, v\} \neq \emptyset$ for all $(u, v) \in E$, then the oscillators achieve synchronization.

Proof: Proposition 1 and Proposition 4, together with the dominating set assumption, imply that synchronization is achieved if all $v \notin A$ satisfy

$$\sum_{u \in N(v)} \alpha_{uv}(x_v) > |\omega_v| \tag{12}$$

for all $x \in [x_v^{final}, x_v^{PI}]$ for some x_v^{PI} such that $|x_v^{PI}| \geq |x_v^{init}|$. This requirement is equivalent to (12) holding with $x_v^{PI} = x_v^{init}$. ■

Proposition 7 implies that the following procedure is sufficient to select input nodes that guarantee synchronization. First, initialize the set of input nodes A to be equal to the set of all nodes v such that Eq. (12) fails. Then, taking the graph $G' = (V', E')$ induced by the vertices $V \setminus A$, find a minimum-size set of nodes A' such that $A' \cap \{u, v\} \neq \emptyset$ for all $(u, v) \in E'$ by solving an instance of the set cover problem. This algorithm will produce an $O(\log n)$ approximation to the minimum-size set satisfying (12) in polynomial-time [37].

V. Input Selection Algorithms for Synchronization

In this section, we propose efficient algorithms for selecting input nodes to guarantee synchronization based on the formulation of Eq. (11). We first present our proposed algorithm, followed by analysis of its optimality bounds.

The input selection algorithm is a two-stage approach. In the first stage, the the set $S \cap \{v_m^i : m = 1, \ldots, M, i = 1, \ldots, MN + 1\}$ is selected in order to approximate the solution to

$$\begin{array}{ll} \text{maximize} & f(S_1) \\ \text{s.t.} & S_1 \in \mathcal{M}_1 \end{array} \tag{13}$$

where $\mathcal{M}_1$ is the matroid consisting of all sets $S \subseteq \tilde{V}$ with $S \in \mathcal{N}_k$ and $S \cap \{v_0 : v \in V\} = \emptyset$. This stage corresponds to selecting a sequence of positive invariant sets that the node states will occupy, with the final state a subset of Λ_{final}. From the previous section, Eq. (13) consists of maximizing a monotone nondecreasing submodular function subject to a partition matroid constraint. A continuous greedy algorithm for solving this problem is described as follows.

The idea of the algorithm is to solve a continuous relaxation of (13) to obtain a solution y, and then round the continuous solution to obtain a discrete set S_1. The continuous problem is formulated as

$$\begin{array}{ll} \text{maximize} & F(y) \\ \text{s.t.} & y \in P(\mathcal{M}_1) \end{array} \tag{14}$$

where $F(y)$ is the multilinear relaxation and $P(\mathcal{M}_1)$ denotes the matroid polytope of $\mathcal{M}_1$. In order to approximate 14), y is initialized to 0. At each iteration, define $a_{v_m^i}$ by

$$a_{v_m^i} = \mathbf{E}(f(R \cup \{v_m^i\}) - f(R)),$$

where R is a random variable generated by sampling each $i \in \tilde{V}$ with probability y_i. A set I is then chosen to satisfy $\max\left\{\sum_{i \in I} a_{v_m^i} : I \in \mathcal{M}_1\right\}$. Since $\mathcal{M}_1$ is a partition matroid, this optimization problem can be solved in linear time. The set y is then incremented as $y(t+\delta) = y(t) + \delta \mathbf{1}_I$, where $\delta = \frac{1}{(N+(NM+1)N)^2}$. The increment is chosen to be $1/\text{rank}(\mathcal{M}_1)^2$, in accordance with [38]. The time index is then incremented to $t + \delta$. The algorithm terminates when $t = 1$.

After the continuous greedy algorithm terminates, the continuous solution y is rounded to obtain a set S using the pipage rounding technique. Since $\mathcal{M}_1$ is a partition matroid, the pipage rounding method simply selects one element in each partition set with probability y_i. A pseudocode description of this approach is given as Algorithm 1.

Algorithm 1 Algorithm for selecting a sequence of positive invariant sets.

procedure FIRST_STAGE_OPT($G = (V, E)$, $\omega_1, \ldots, \omega_n$, $\{x_v^{max} : v \in V\}$, γ, $\{\overline{x}_v : v \in V\}$)
  Input: Graph $G = (V, E)$, frequencies $\omega_1, \ldots, \omega_n$, input nodes A, bounds $\{x_v^{max}\}$ on each node state, desired level of synchronization γ, bounds $\overline{x}_v$ on initial node states.
  Output: Set S_1
  $\delta \leftarrow \frac{1}{(N+(NM+1)N)^2}$
  for $k = 1, \ldots, (N + (NM + 1) * N)^2$ **do**
    for $v \in V$, $i = 0, \ldots, MN + 1$, $m = 1, \ldots, M$ **do**
      $a_{v_m^i} \leftarrow \mathbf{E}(f(R \cup \{v_m^i\})) - \mathbf{E}(f(R))$
    end for
    for $v \in V$, $i = 0, \ldots, (MN + 1)$ **do**
      $I \leftarrow I \cup \{\arg\max\{a_{v_m^i} : m = 1, \ldots, M\}\}$
    end for
    $y \leftarrow y + \delta \mathbf{1}_I$
  end for
  $S_1 \leftarrow \emptyset$
  for $v \in V$, $i = 0, \ldots, MN + 1$ **do**
    Choose exactly one $\tilde{v}_m^i$ from $\{v_m^i : m = 0, \ldots, M\}$ with probability y_m^i
    $S_1 \leftarrow S_1 \cup \{\tilde{v}_m^i\}$
  end for
  return S_1
end procedure

The complexity of Algorithm 1 is described by the following lemma.

Lemma 6: Algorithm 1 terminates in $O(N^6 M^4)$.

Proof: The rank of the matroid is equal to $N(MN + 1)$, implying that there are $O(M^2 N^4)$ iterations of the outer loop. At the inner loop, the cost is dominated by computing the increments $\mathbf{E}(f(R \cup \{v_m^i\})) - \mathbf{E}(f(R))$. This leads to an overall complexity of $O(N^6 M^4)$. ■

The optimality bounds provided by this approach are summarized as follows.

Lemma 7: Algorithm 1 returns a set S satisfying $f(S) \geq (1 - 1/e) f(S^*)$, where S^* is the optimal solution to (13).

Proof: From [38], the continuous greedy algorithm is guaranteed to select an approximate solution S to

$\max\{f(S) : S \in \mathcal{M}\}$, where $f(S)$ is a monotone submodular function and $\mathcal{M}$ is a matroid, satisfying

$$f(S) \geq (1 - 1/e) f(S^*)$$

where S^* is the optimal solution. The lemma then follows from monotonicity and submodularity of $f(S)$ and the matroid structure of $\mathcal{M}_1$. ■

In addition, the complexity of the algorithm can be reduced, at the cost of some optimality, by using a simple greedy approach. Under the greedy algorithm, at each iteration, the element $\{v_m^i\}$ is chosen that maximizes $f(S \cup \{v_m^i\})$, terminating when S is a basis of $\mathcal{M}_1$. This algorithm has $O(MN^2(NM(NM+1))) = O(N^4 M^3)$ complexity, and an optimality bound of $1/2$.

After the set S_1 has been chosen, the next stage is to choose an input set S_0 that ensures $f(S_0 \cup S_1) \geq \alpha$. This second stage optimization problem is written as

$$\begin{array}{ll} \text{minimize} & |S_0| \\ \text{s.t.} & f(S_0 \cup S_1) \geq \zeta \end{array} \quad (15)$$

where

$$\zeta = \sum_{v \in V} \tau_v + \sum_{i=1}^{MN+1} \sum_{v \in V} \beta_v.$$

A simple greedy algorithm is sufficient to approximate (15) up to a provable optimality bound. At each iteration of the algorithm, the element $\{v_0\}$ that maximizes $f(S_0 \cup S_1 \cup \{v_0\})$ is added to the set, terminating when $f(S_0 \cup S_1) \geq \zeta$. A pseudocode description is given as Algorithm 2.

Algorithm 2 Algorithm for selecting a set of input nodes A to ensure convergence to Λ_{final}.

procedure SECOND_STAGE_GREEDY($G = (V, E)$, $\omega_1, \ldots, \omega_n$, $\{x_v^{max} : v \in V\}$, γ, $\{\overline{x}_v : v \in V\}$, S_1)
  Input: Graph $G = (V, E)$, frequencies $\omega_1, \ldots, \omega_n$, input nodes A, bounds $\{x_v^{max}\}$ on each node state, desired level of synchronization γ, bounds $\overline{x}_v$ on initial node states, set S_1.
  Output: Input set A
  $S_0 \leftarrow \emptyset$
  while $f(S_0 \cup S_1) < \zeta$ **do**
    for $v \in V$ **do**
      $\lambda_v \leftarrow f(S_0 \cup S_1 \cup \{v_0\})$
    end for
    $v^* \leftarrow \arg\max \lambda_v$
    $S_0 \leftarrow S_0 \cup \{v_0^*\}$
  end while
  $A \leftarrow \{v : v_0 \in S_0\}$
  return A
end procedure

Algorithm 2 has complexity $O(N^2)$, and hence the overall complexity is dominated by the cost of selecting a positive invariant set. The following proposition describes the optimality guarantees of Algorithm 2.

Proposition 8: Algorithm 2 selects a set S_0 satisfying

$$\frac{|S_0|}{|S_0^*|} \leq 1 + \log\left\{\frac{\sum_{v \in V} |\omega_v| + \epsilon}{\sigma^*}\right\}$$

where S_0^* is the optimal solution to (15) and

$$\sigma^* = \min\{\sigma_{uv}(0, m) - \sigma_{uv}(m', m) : (u, v) \in E, m = 1, \ldots, M, m' = 1, \ldots, M\}.$$

Proof: From [39], the greedy algorithm is known to select a set S satisfying

$$\frac{|S|}{|S^*|} \leq 1 + \log\left\{\frac{f(V) - f(\emptyset)}{f(S) - f(S_{t-1})}\right\},$$

where S_{t-1} is the set at the second-to-last iteration of the greedy algorithm. In this case,

$$f(V) - f(S_1) \leq \sum_{v \in V} (|\omega_v| + \epsilon),$$

while the last increment is bounded below by

$$\sigma_{uv}(0, m) - \sigma_{uv}(m', m)$$

where u is the last input node added to the set and $u_{m'} \in S_1$. Taking the minimum over all possible values of u, v, m, and m' as a lower bound gives the desired result. ■

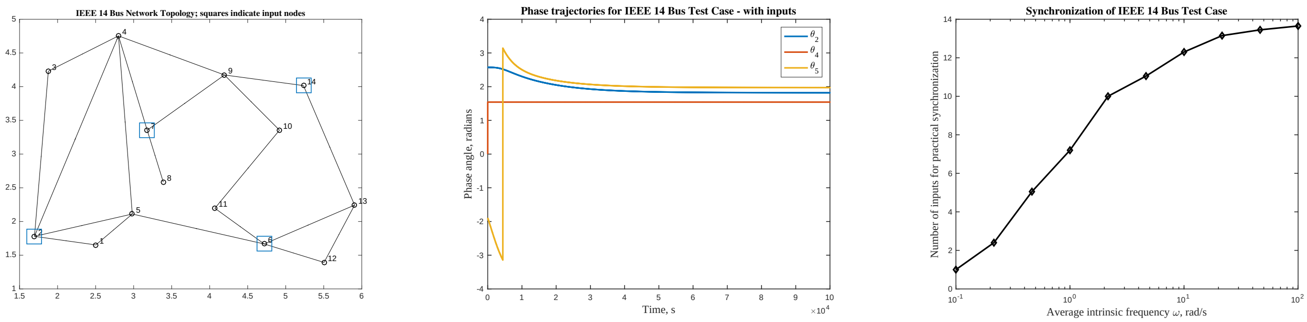
The bound in Proposition 8 depends on the functions α_{uv}. An analysis for the special case of the Kuramoto model is as follows. In this case, $\alpha_{uv}(x_v - x_u) = \Gamma_{uv} \sin(x_v - x_u)$ for some coupling coefficient $\Gamma_{uv} \geq 0$. For simplicity, assume that all coupling coefficients are equal to the same value Γ. Then for M sufficiently large, the minimum value is approximately equal to $\frac{\Gamma}{M}$, giving an upper bound of

$$\frac{|S|}{|S^*|} \leq 1 + \log \frac{M}{\Gamma} \sum_{v \in V} (|\omega_v| + \epsilon).$$

VI. Numerical Results

We investigated our approach through a numerical study. The goal of our study was to analyze the impact of the external inputs on the node dynamics, as well as to investigate the properties of the inputs chosen by our algorithm. In order to reduce the computation time, we used the greedy approximation to the optimization problem (13). We considered synchronization of phase angles in the IEEE 14 Bus power system test case [17]. The parameters $M = N = 20$. The node coupling values were chosen as $\alpha_{uv}(x_v - x_u) = -\Gamma_{uv} \sin(x_v - x_u)$ for some $\Gamma_{uv} \geq 0$, i.e., the Kuramoto model [6].

Synchronization plays a vital role in the power system, in which stable operation requires buses and generators to maintain the same frequency and a relative phase difference of no more than $\pi/2$ on each transmission line (edge) [40]. In order to evaluate our approach to synchronization of power systems, we consider the first-order model with negligible resistance on the transmission lines. The input nodes represent generators that are fixed to a common reference phase and frequency in order to restore stability to the grid. We studied phase synchronization in the power system using the IEEE 14 Bus case study [17], which provides a network topology with $n = 14$ nodes along with the line impedances of the edges. We defined the coupling coefficient $\Gamma_{uv} = 1/\eta_{uv}$, where η_{uv} is the magnitude of the impedance of edge (u, v). The intrinsic frequencies were chosen according to a zero-mean Gaussian



(a) (b) (c)

Fig. 2. Synchronization case study using the IEEE 14 Bus case study [17]. Coupling coefficients were chosen to be equal to the inverse of the magnitudes of the line impedances, while the intrinsic frequencies were chosen from a Gaussian distribution. (a) Illustration of 14-bus network and inputs needed for one trial when the variance of the intrinsic frequencies was equal to 8. The black squares indicate input nodes. (b) Sample trajectories for nodes 2, 4, and 5 in the 14-bus network. Node trajectories converge to within the desired phase difference ($\gamma = \pi/3$) of each other. (c) Number of inputs required for synchronization as a function of the variance of the intrinsic frequencies. As the variance increases, the frequencies of neighboring nodes diverge and hence more inputs are needed to ensure synchronization.

distribution with variance ranging from 0.01 to 100 in different trials. The goal was to ensure that the phase angles satisfied $|x_v(t)| < \frac{\pi}{5}$ asymptotically, while ensuring that $|x_v(t)| \leq \pi/4$ for all t, consistent with the goal of guaranteeing power system stability [40].

Figure 2(a) illustrates input selection for a given set of intrinsic frequencies. The variance of the intrinsic frequencies was equal to 8. The larger squares indicate input nodes selected by our algorithm. We observe that each non-input node is at most two hops away from an input node, which suggests that centrally located nodes are more likely to be candidates for inputs. On the other hand, nodes with high degree were not necessarily chosen as inputs. Additional information beyond the network topology, including the intrinsic frequencies and the coupling coefficients between nodes, is incorporated into the input selection process.

In Figure 2(b), trajectories of non-input nodes 2, 4, and 5 from Figure 2(a) are shown. The trajectories of these neighboring nodes converge to within the desired bound of $\frac{\pi}{3}$ from each other, and synchronize to the frequency of the input nodes. We selected $\omega_0 = 0$ for clarity of the figure; as discussed in Section III-A, we can make this assumption without loss of generality.

Figure 2(c) shows the number of input nodes required to achieve synchronization for different intrinsic frequencies. When the variance of $\{\omega_v : v \in V\}$ is low, only one input node is required to achieve synchronization. For variance of 1, on average 7.2 inputs are required for synchronization, while for large variance in the intrinsic frequencies all nodes must be selected as input nodes to guarantee synchronization.

VII. Conclusions and Future Work

We considered the problem of ensuring synchronization in networks with nonlinear dynamics by introducing external input nodes. We proved that synchronization is achieved if two conditions hold. First, if there exists a positive invariant set Λ_{PI} that contains the set of initial node states and lies within a set of upper bounds on the node phases. Second, for any initial state inside Λ_{PI}, each node's phase is guaranteed to converge to within the desired level of synchronization.

We formulated a submodular optimization approach for selecting a set of input nodes to guarantee synchronization. Under the submodular optimization approach, each condition is mapped to a threshold condition on a monotone submodular function, together with a partition matroid constraint. Based on the submodular formulation, we proposed efficient algorithms with provable optimality bounds for selecting a set of up to k nodes in order to maximize the level of synchronization in the network, as well as selecting the minimum-size input set to guarantee a desired level of synchronization.

Our approach was validated through a numerical study of synchronization in power systems. Our numerical study supports the intuition that the number of input nodes required decreases as the coupling between neighboring nodes grows stronger. In addition, we found that centrally located nodes are often chosen as inputs, so that the maximum distance between a node and the input set is small.

The threshold-based conditions derived in this work are sufficient, but not necessary. Characterizing the space of networks where these conditions are also necessary, as well as developing tighter sufficient conditions, remains an open problem. Furthermore, while the model we studied considers the first-order dynamics of the nodes, second-order dynamics are often used to model systems such as the power grid [23]. Generalizing our approach to these second-order systems is a direction of future research.

Appendix A
Proofs

In this section, we provide proofs of Lemma 3, Proposition 3, and Proposition 6. For ease of exposition, we divide the proofs into a sequence of lemmas. We begin with a preliminary result.

Lemma 8: For any r with $0 \leq r \leq 2M$,

$$K_{uv} \log \frac{r+2}{r+1} \geq Z_{uv} + K_{uv} \log \frac{r+3}{r+2}. \qquad (16)$$

Proof: By definition, $K_{uv} = \frac{Z_{uv}}{\log \phi}$ with $\phi \in \left[1, 1+\frac{1}{4M^2+8M+3}\right]$. Eq. (16) is therefore equivalent to

$$K_{uv}\log\frac{r+2}{r+1} \geq K_{uv}\left(\log\phi + \log\frac{r+3}{r+2}\right).$$

Rearranging terms implies that (16) is equivalent to

$$\frac{r+2}{r+1} \geq \phi\frac{r+3}{r+2}.$$

By definition of ϕ,

$$\begin{aligned}\phi\frac{r+3}{r+2} &\leq \frac{r+3}{r+2}\left(1+\frac{1}{4M^2+8M+3}\right)\\ &\leq \frac{r+3}{r+2}\left(1+\frac{1}{r^2+4r+3}\right)\\ &= \frac{r+3}{r+2}\left(\frac{(r+2)^2}{(r+1)(r+3)}\right) = \frac{r+2}{r+1},\end{aligned}$$

completing the proof. ∎

We now establish monotonicity of $f_v(S)$.

Proof of Lemma 3: Consider the increment $f_v(S \cup \{w_l\}) - f_v(S)$ for some $w_l \in \overline{V} \setminus S$. We divide the proof into four cases, namely, (i) $w_l = v_0$, (ii) $w \notin \{u, v\}$, (iii) $w = u_0$, and (iv) $w \in \{u_m : m = \overline{m}_u, \ldots, M\} \cup \{v_m : m = \overline{m}_v, \ldots, M\}$.

Case (i): By construction, if $v_0 \in S$, then $f_v(S) = R$ for some R with $R > K_v \log 2M + Z_v$, i.e., R is an upper bound on $f_v(S)$ for any S with $v_0 \notin S$. Hence the increment is positive.

Case (ii): If $w \notin \{u, v\}$, then $f_v(S \cup \{w_l\}) = f_v(S)$.

Case (iii): If $v_0 \in S$, then $f_v(S \cup \{u_0\}) - f_v(S) = 0$. Otherwise, since adding u_0 to S does not change the set S_{uv}, we have

$$\begin{aligned}f_v(S) &= \sum_{u \in N(v)}\Bigg[K_{uv}\log(|S_{uv}|+1)\\ &\quad + \sum_{u_{m'}, v_m \in S}\rho_{uv}(m', m)\Bigg]\\ f_v(S \cup \{u_0\}) &= \sum_{u \in N(v)}\Bigg[K_{uv}\log(|S_{uv}|+1)\\ &+ \max\left\{\sum_{u_{m'}, v_m \in S}\rho_{uv}(m', m), \min_{v_m \in S}\rho(0, m)\right\}\Bigg],\end{aligned}$$

and hence

$$\begin{aligned}&f_v(S \cup \{u_0\}) - f_v(S)\\ &= \sum_{u \in N(v)}\Bigg[\max\left\{\sum_{u_{m'}, v_m \in S}\rho_{uv}(m', m), \min_{v_m \in S}\rho(0, m)\right\}\\ &\qquad - \sum_{u_{m'}, v_m \in S}\rho_{uv}(m', m)\Bigg],\end{aligned}$$

which is nonnegative.

Case (iv): We have that

$$\begin{aligned}f_{uv}(S \cup \{w_l\}) &\geq K_{uv}\log(|S_{uv}|+2)\\ &\geq K_{uv}\log(|S_{uv}|+1) + Z_{uv}\\ &\quad + K_{uv}\log\frac{|S_{uv}|+3}{|S_{uv}|+2}\\ &\geq K_{uv}\log(|S_{uv}|+1) + Z_{uv} \geq f_{uv}(S),\end{aligned}$$

where (17) follows from Lemma 8. Summing over $u \in N(v)$ yields the desired result. ∎

We now prove submodularity of $f_v(S)$. To show submodularity of $f_v(S)$, we need to establish that, for any $S \subseteq T$ with $w_l \notin T$,

$$f_v(S \cup \{w_l\}) - f_v(S) \geq f_v(T \cup \{w_l\}) - f_v(T).$$

For clarity, we consider three cases of w_l as separate lemmas, namely (a) $w_l = v_0$ (Lemma 9), (b) $w_l = u_0$ (Lemma 10), and (c) $w_l \in \{u_m : m = \overline{m}_u, \ldots, M\} \cup \{v_m : m = \overline{m}_v, \ldots, M\}$ (Lemma 11). The results are then summarized in a proof of Proposition 3.

Lemma 9: For any $S \subseteq T$ with $v_0 \notin T$,

$$f_v(S \cup \{v_0\}) - f_v(S) \geq f_v(T \cup \{v_0\}) - f_v(T).$$

Proof: By definition of f_v, $f_v(S \cup \{v_0\}) = f_v(T \cup \{v_0\}) = R$. The desired result is therefore equivalent to $f_v(S) \leq f_v(T)$, which holds by Lemma 3. ∎

We next prove submodularity when u_0 is added to the set S.

Lemma 10: For any $S \subseteq T$ with $u_0 \notin S$,

$$f_v(S \cup \{u_0\}) - f_v(S) \geq f_v(T \cup \{u_0\}) - f_v(T). \tag{17}$$

Proof: Since S_{uv} and T_{uv} are left unchanged by the addition of $\{u_0\}$, we have that

$$\begin{aligned}&f_{uv}(S \cup \{u_0\}) - f_{uv}(S)\\ &\quad = \max\left\{\sum_{\substack{u_{m'} \in S\\ v_m \in S}}\rho_{uv}(m', m), \min_{v_m \in S}\rho(0, m)\right\}\\ &\qquad\qquad - \sum_{\substack{u_{m'} \in S\\ v_m \in S}}\rho_{uv}(m', m).\end{aligned}$$

We consider four sub-cases. In the first case,

$$\sum_{\substack{u_{m'} \in S\\ v_m \in S}}\rho_{uv}(m', m) \geq \min_{v_m \in S}\rho_{uv}(0, m)$$
$$\sum_{\substack{u_{m'} \in T\\ v_m \in T}}\rho_{uv}(m', m) \geq \min_{v_m \in T}\rho_{uv}(0, m)$$

When these conditions hold, $f_{uv}(S \cup \{u_0\}) = f_{uv}(S)$ and $f_{uv}(T \cup \{u_0\}) = f_{uv}(T)$. In the second case,

$$\sum_{\substack{u_{m'} \in S\\ v_m \in S}}\rho_{uv}(m', m) \geq \min_{v_m \in S}\rho_{uv}(0, m)$$
$$\sum_{\substack{u_{m'} \in T\\ v_m \in T}}\rho_{uv}(m', m) < \min_{v_m \in T}\rho_{uv}(0, m)$$

These conditions, however, cannot hold simultaneously. If the first inequality holds, then

$$\min_{v_m \in T} \rho_{uv}(0,m) \le \min_{v_m \in S} \rho_{uv}(0,m) \le \sum_{\substack{u_{m'} \in S \\ v_m \in S}} \rho_{uv}(m',m) \le \sum_{\substack{u_{m'} \in T \\ v_m \in T}} \rho_{uv}(m',m),$$

contradicting the second inequality.

In the third case,

$$\sum_{\substack{u_{m'} \in S \\ v_m \in S}} \rho_{uv}(m',m) < \min_{v_m \in S} \rho_{uv}(0,m)$$
$$\sum_{\substack{u_{m'} \in T \\ v_m \in T}} \rho_{uv}(m',m) \ge \min_{v_m \in T} \rho_{uv}(0,m)$$

We then have $f_{uv}(S \cup \{u_0\}) > f_{uv}(S)$ and $f_{uv}(T \cup \{u_0\}) = f_{uv}(T)$, implying that submodularity holds in this case as well. In the final case,

$$\sum_{\substack{u_{m'} \in S \\ v_m \in S}} \rho_{uv}(m',m) < \min_{v_m \in S} \rho_{uv}(0,m)$$
$$\sum_{\substack{u_{m'} \in T \\ v_m \in T}} \rho_{uv}(m',m) < \min_{v_m \in T} \rho_{uv}(0,m)$$

We have that

$$f_{uv}(S \cup \{u_0\}) - f_{uv}(S) = \min_{v_m \in S} \rho_{uv}(0,m) - \sum_{\substack{u_{m'} \in S \\ v_m \in S}} \rho_{uv}(m',m)$$

Hence

$$\min\{\rho_{uv}(0,m) : v_m \in S\} \le \min\{\rho_{uv}(0,m) : v_m \in T\},$$

since $S \subseteq T$. Similarly,

$$\sum_{\substack{u_{m'} \in S \\ v_m \in S}} \rho_{uv}(m',m) \le \sum_{\substack{u_{m'} \in T \\ v_m \in T}} \rho_{uv}(m',m).$$

These inequalities imply that Eq. (17) holds. ■

The next lemma treats the case where v_m or u_m is added to S for some $m \neq 0$.

Lemma 11: For any $S \subseteq T$ and $w_l \notin T$, with

$$w_l \in \{u_m : m = \overline{m}_u, \dots, M\} \cup \{v_m : m = \overline{m}_v, \dots, M\},$$

we have that

$$f_v(S \cup \{w_l\}) - f_v(S) \ge f_v(T \cup \{w_l\}) - f_v(T).$$

Proof: The increment $f_{uv}(S \cup \{w_l\}) - f_{uv}(S)$ satisfies

$$\begin{aligned} f_{uv}(S \cup \{w_l\}) - f_{uv}(S) &\ge \log \frac{|S_{uv}|+2}{|S_{uv}|+1} && (18) \\ &\ge Z_{uv} + \log \frac{|S_{uv}|+3}{|S_{uv}|+2} && (19) \\ &\ge Z_{uv} + \log \frac{|T_{uv}|+2}{|T_{uv}|+1} && (20) \\ &\ge f_{uv}(T \cup \{w_l\}) - f_{uv}(T), && (21) \end{aligned}$$

where (19) follows from Lemma 8. ■

Combining the above lemmas leads to the proof of Proposition 3.

Proof of Proposition 3: First, note that if $w \notin \{u, v\}$, then $f_v(S \cup \{w_l\}) = f_v(S)$ and $f_v(T \cup \{w_l\}) = f_v(T)$. The other three cases of w_l are treated by Lemmas 9–11. ■

The proof of Proposition 6 follows the proofs of Lemma 3 and Proposition 6.

Proof of Proposition 6: The proof of monotonicity of each $f_{uv}^i(S)$ is similar to the proof of Lemma 3, using the definition of Z_{uv} from Section IV-C. The proof of submodularity of $f_{uv}^i(S)$ consists of four different cases of

$$f_{uv}^i(S \cup \{w_l^j\}) - f_{uv}^i(S) \ge f_{uv}^i(T \cup \{w_l^j\}) - f_{uv}^i(T),$$

namely, (a) $w_l^j = v_0$, (b) $w_l = u_0$, (c) $w_l^j \in \{u_m^i : m = 1, \dots, M\} \cup \{v_m^i : i = 1, \dots, M\}$, and (d) any other value of w_l^j. The proof of submodularity in Case (a) is analogous to the proof of Lemma 9. Case (b) is analogous to the proof of Lemma 10, and case (c) is similar to the proof of Lemma 11. The increment is left unchanged in case (d), implying that submodularity holds. ■

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