

Minimal Input and Output Selection for Stability of Systems with Uncertainties

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Abstract—In networked control systems, selecting a subset of input and output nodes is a crucial step in designing a stabilizing controller. Most existing approaches to input and output selection focus on nominal systems with known parameters. For systems with uncertainties and time delays, current selection methods are based on exploiting the convexity after relaxing the original problem, which is inherently discrete, to continuous optimization forms and hence lack optimality guarantees. This paper studies the problem of identifying the minimum-size sets of input and output nodes to guarantee stability of a linear system with uncertainties and time delays. We derive sufficient conditions to guarantee existence of a stabilizing controller for an uncertain linear system, based on a subset of system modes lying within the controllability and observability subspaces induced by the selected inputs and outputs. We then formulate the problems of selecting minimum-size sets of input and output nodes to satisfy the derived conditions, and prove that they are equivalent to discrete optimization problems with bounded submodularity ratios. We develop polynomial-time selection algorithms with provable guarantees on the minimum number of inputs and outputs required. Our approach is applicable to various types of uncertainties, including additive uncertainty, multiplicative uncertainty, uncertain output delay, and structured uncertainty. In a numerical study, we test our approach for the wide-area damping control in power systems to ensure small signal stability. Our results are validated on the IEEE 39-bus test power system.

Index Terms—Input selection, output selection, robust control, submodularity ratio, optimization.

I. INTRODUCTION

A broad class of complex dynamical systems, including power systems, neural circuits, and traffic dynamics, can be described as networked control systems [1]. Such systems are often controlled by sending control signals to a selected subset of input nodes while observing the state values at selected output nodes. Examples of this approach include control and estimation of power systems via generators and phasor measurement units (PMUs), as well as leader-follower multi-agent systems. The choice of input and output nodes is known to determine performance criteria including stability, robustness to noise, controllability, and observability of the system [2], [3], [4], [5], [6].

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Designing a controller coordinating a large set of input and output nodes is often impractical, which motivates the need for minimal input and output selection mechanisms. The problem of identifying a minimum-size set of input/output nodes in a large-scale networked system to satisfy desired control requirements has attracted research interest in recent years. Various algorithms have been proposed to address this problem with different selection criteria, including controllability [2], [3], reachability [4], structural controllability [5], and robustness to noise [6], as well as application-specific methodologies for power systems [7], [8].

Most existing approaches to the minimal input/output selection problem assume that no uncertainty or time delay is present in the system, and consider one or more of the following system models: linear systems with known, fixed parameters [2], systems with a discrete set of parameter values [6], or systems with arbitrary real-valued parameters [5]. At present, there has been little study of input or output selections in the presence of continuous system uncertainties, such as small perturbations in system matrices. It has been shown that inputs and outputs that are selected based on a nominal system model may fail to guarantee the system stability with these uncertainties present [9]. Current works addressing the minimal selection problem for systems with uncertainties are based on relaxing the inherently combinatorial problem to continuous convex optimization form. This approach does not guarantee the optimality of solution.

This paper presents an approach to minimal input and output selection that guarantees existence of a stabilizing feedback controller in the presence of delays and uncertainties of different types. By exploiting the submodularity ratio of this combinatorial problem, we establish computationally efficient approximation algorithms with provable optimality guarantees. Our key insights are the following. First, we show that it is sufficient to guarantee the existence of such a stabilizing controller if a subset of system modes, determined by the H_∞ norm of the uncertain perturbation, have eigenvectors lying in both the controllability and observability matrix subspaces induced by the selected input and output nodes, respectively. Second, we show that the problem of selecting the minimum-size sets of input and output nodes that satisfy the above controllability and observability conditions is equivalent to a column-subset selection problem with bounded submodularity ratio [10]. We make the following specific contributions:

- We derive a sufficient condition for a set of input nodes and a set of output nodes to guarantee the existence of a feedback controller that can stabilize a given uncertain

linear system. This condition is based on the fact that a set of system modes is required to lie in the controllability and observability matrix subspace resulting from the input and output nodes. We extend our results to systems with uncertain output delays, as well as systems with structured uncertainties.

- We formulate the problem of selecting minimum-cardinality sets of input and output nodes to satisfy the derived conditions. We prove that the input and output selection problems are equivalent to minimizing the Euclidean distance between eigenvectors of the identified system modes and the controllability/observability subspace. We also prove that these distances are monotone decreasing functions of the selected input/output sets with bounded submodularity ratio.
- We propose polynomial-time approximation algorithms for selecting minimum-cardinality sets of input and output nodes guaranteeing system stabilizability. We characterize the optimality gap/bound of our approach via the submodularity ratio, which we derive as a function of the system parameters. We analyze different types of uncertainties under our framework, including uncertain delays and structured uncertainties.
- We evaluate our approach through a numerical study on the IEEE 39-bus power system [11]. From the results, we conclude that our proposed input/output selection approach better ensures the system robustness to uncertainties, compared to selections that do not take uncertainties into account. With the same perturbation, the proposed algorithm with robustness consideration results in a system that is stable in a higher percentage of cases compared to the selection without considering uncertainties. In comparison to a geometric indices based input/output selection approach, our method requires fewer input and output nodes to satisfy the stability conditions.

The paper is organized as follows. Section II reviews the related work. Section III introduces the system model. Section IV presents the proposed approach to input and output selections. Section V contains analysis for different types of uncertainties. Section VI presents numerical results. Section VII concludes the paper.

II. RELATED WORK

Input and output selection ensuring stability-related properties is a critical step in controlling networked systems and hence has received extensive research attention over decades. Early approaches were summarized in [12], where various selection criteria were discussed, including controllability/observability, frequency-dependent minimum singular value, and condition number. These methods, however, cannot scale to large-scale systems that have a nearly unlimited number of interacting parts, such as biological systems and the smart grid. Initial studies of identifying a minimum-size set of input nodes to control a large-scale complex network were presented in [13].

In recent years, a series of works have been proposed to address the problem of selecting the minimum-size set of

inputs for linear control systems with a diagonal input matrix while satisfying certain controllability requirements, known as the minimal controllability problem. This problem was first presented in [2], where greedy algorithms were proposed to approximate the optimal solution with provable guarantees on sparsity. The exact solution to the minimal controllability problem was studied in [3], where authors show that this problem can be reduced to the minimum set covering problem.

Most existing works on input and output selection focus on selecting a minimum-size set of nodes to satisfy one or more performance criteria [5], [6], [14], [15], [16], [17]. In [14], the authors studied the problem of selecting a minimum-size set of inputs to ensure controllability while the average minimum control effort for stabilizing the system is within a bound. In [5], the constraint is to ensure structural controllability, defined by the existence of a controllable numerical realization of the linear system with the same structure (i.e., zero/nonzero patterns) in system matrices. This problem was extended in [16] by considering output selection and an additional constraint on the communication cost associated with feeding outputs to inputs. The submodular structures of the minimal input selection problems with different constraints were exploited in [6] and [15], and efficient selection algorithms have been proposed to approximate each problem with provable optimality bounds. The joint problem of selecting control nodes and designing controller actions is addressed in [17]. All these approaches select minimal sets of input nodes to ensure the controllability of a nominal system, while our approach focuses on the stabilizability of uncertain systems, defined by the ability of a system to reach an equilibrium.

Controllability is a sufficient but not necessary condition to ensure system stability in many control systems. Controllability implies that all system modes are controllable, while stabilizability only requires that a subset of unstable modes be controllable [18]. The problem of selecting a minimal set of inputs to achieve stabilizability was studied in [4], where approximation algorithms were proposed to greedily select input nodes until the desired modes are contained in the controllability matrix subspace. These works assume known, fixed system parameters and have not taken into account plant uncertainties or time delays.

Uncertainties and time delays are common problems in linear systems. In [9], the authors have proposed a heuristic approach towards the input and output selection problem guaranteeing system robustness to uncertainties. More recent approaches to minimal input and output selection for systems with uncertainties rely on relaxing the discrete combinatorial problem to continuous convex optimization forms. For example, the sparsity-promoting method using l_1 -norm [7], the H_2/H_∞ optimization approach for centralized control systems [19], and the minimal input selection for decentralized control systems [20]. These heuristic methods, however, do not provide guarantees on the optimality of solutions of the initial problem. In this paper, we propose a bounded submodularity ratio approach with provable optimality guarantees to the minimal input and output selection problem that ensures the existence of a robust control and achieves system reachability from a given state to the origin.

The difficulty of approximating minimal reachability problems is presented in [21], which proves that there is no polynomial-time algorithm to achieve a set of inputs within a constant factor of true optimal selections. In this paper, we show that the bounded submodularity ratio alone is sufficient to achieve certain approximations, even without submodularity or supermodularity. The optimality bound, however, can be arbitrarily small for certain system parameters and hence is consistent with [21].

Preliminary versions of this work appeared in [22] for nominal systems and [23] for uncertain systems. The optimality bounds reported in [22], [23] were derived based on a result from the existing literature on submodularity [24] which has been proven to be incorrect in [21]. A corrected version of our preliminary results appears in [25]. In this journal version, we add new approaches to output selection and analysis for structured uncertainty cases, and also provide a detailed numerical validation of the proposed selection algorithms.

III. SYSTEM MODEL AND PRELIMINARIES

In this section, we present our system model and introduce notations that will be used throughout the paper. We first describe the generalized linear system with uncertainties and then show a transformation to the $M - \Delta$ loop system. We also give backgrounds on the submodularity ratio.

A. Uncertain Linear Systems

We consider a linear system with a perturbation $\delta(t)$ in system dynamics, described by

$$\dot{x}(t) = Ax(t) + Bu(t) + \delta(t) \quad (1)$$

$$y(t) = Cx(t), \quad (2)$$

where $x \in \mathbb{R}^n$ is the vector of n states, $y \in \mathbb{R}^m$ is the output vector, and $u \in \mathbb{R}^p$ is the input vector. The i th entries of u and y are denoted by u_i and y_i , representing an input control signal and an output observation, respectively. Matrices A , B and C are constant and given, where $A \in \mathbb{R}^{n \times n}$ has eigenvalues $\{\lambda_i\}$ and corresponding eigenvectors $\{v_i\}$.

The vector $\delta \in \mathbb{R}^n$ represents an unknown perturbation affecting the system dynamics and the resulting system (1)-(2) is an uncertain system. For many types of perturbations, the uncertain term δ can be expressed as a linear function of the state x and an unknown matrix Δ whose singular values are bounded by $\|\Delta\|_\infty \leq \sigma$. For example, when an additive perturbation occurs in matrix A and results in a system $\dot{x}(t) = (A + \Delta)x(t) + Bu(t)$, the uncertain term is represented by $\delta(t) = \Delta x(t)$.

Eq. (1)-(2) generalize both uncertain linear systems and certain linear systems. A system with no uncertainty will have $\delta(t) = 0$ for all t , or equivalently, $\|\Delta\|_\infty = 0$ assuming $\delta(t) = \Delta x(t)$.

Given a linear system (1)-(2), the input matrix $B \in \mathbb{R}^{n \times p}$ has columns $\{b_1, \dots, b_p\}$ where each column b_i corresponds to the influence of a control input u_i to the system, while the output matrix $C \in \mathbb{R}^{m \times n}$ has rows $\{c_1, \dots, c_m\}$ where each row c_i represents the connection between an output y_i and the states x .

Let $\Omega_{in} = \{1, \dots, p\}$ be the set of indices of all inputs and let $\Omega_{out} = \{1, \dots, m\}$ be the set of indices of all outputs. Selecting a subset $S \subseteq \Omega_{in}$ of inputs and a subset $T \subseteq \Omega_{out}$ of outputs changes the system dynamics to

$$\dot{x}(t) = Ax(t) + B_S u_S(t) + \delta(t) \quad (3)$$

$$y_T(t) = C_T x(t), \quad (4)$$

where the input vector u_S has entries $\{u_i : i \in S\}$; the matrix B_S has columns $\{b_i : i \in S\}$; the output vector y_T has entries $\{y_i : i \in T\}$; and the matrix C_T has rows $\{c_i : i \in T\}$. The input and output selections therefore impact the system dynamics by determining the columns of matrix B and the rows of matrix C , respectively.

The goal of the input and output selection is to ensure the existence of a feedback control u_S based on output y_T such that the system (3)-(4) can be driven from the initial state $x(t_0)$ at time t_0 to the origin at some time t_1 in the presence of uncertainty $\delta(t)$. Such control u_S is called a stabilizing controller. In our approach, we consider the following design of such a controller.

Let $\hat{x}(t)$ be an estimate of the system state $x(t)$ with dynamics

$$\dot{\hat{x}}(t) = A\hat{x}(t) + B_S u_S(t) + L(y(t) - C_T \hat{x}(t)),$$

where L is a weighting matrix on the output error $(y(t) - C_T \hat{x}(t))$ that helps correct the estimation dynamics. Defining the estimation error $e(t) = x(t) - \hat{x}(t)$ induces the following dynamics

$$\dot{e}(t) = (A - LC_T)e(t) + \delta(t). \quad (5)$$

We consider a feedback control based on the state estimation, defined by

$$u_S(t) = -K\hat{x}(t), \quad (6)$$

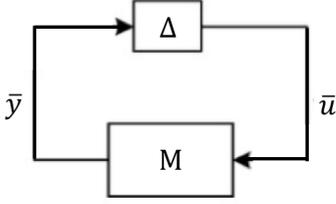
where K is the control matrix to be designed. Then by involving the feedback control (6) and the state estimation dynamics (5), the closed-loop dynamics of system (3)-(4) is given by

$$\begin{bmatrix} \dot{\hat{x}}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} A - B_S K & B_S K \\ 0 & A - LC_T \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} I \\ I \end{bmatrix} \delta(t). \quad (7)$$

The system (7) is asymptotically stable if there exists a feedback matrix K and a state-estimate matrix L such that the initial state $[x(t_0), e(t_0)]$ at time t_0 can be driven to zero at some time t_1 in the presence of uncertainty $\delta(t)$. Such a pair of matrices (K, L) gives a stabilizing controller.

B. $M - \Delta$ Loop Representation

A system in $M - \Delta$ loop has structures as shown in Fig. 1, where the block Δ is an uncertain matrix with singular values bounded by $\|\Delta\|_\infty \leq \sigma$, while the block M is the nominal dynamics of the system. The vector \bar{y} represents the parameters in system M that incurs perturbation and \bar{u} represents the impact of the uncertainty Δ into the system M . They are related by $\bar{u} = \Delta \bar{y}$.

Fig. 1: M- Δ loop for stability analysis.

The block M has a state-space realization described by the following equations

$$\dot{\bar{x}} = A_{cl}\bar{x} + B_{cl}\bar{u} \quad (8)$$

$$\bar{y} = C_{cl}\bar{x}, \quad (9)$$

where \bar{x} is the system state and A_{cl}, B_{cl}, C_{cl} are known constant matrices.

The closed-loop system (7) can be transformed into an $M - \Delta$ loop system for stability analysis [26]. The parameters in (8)-(9) relate to those in Eq. (7) by the following equations

$$\bar{x} = \begin{bmatrix} x \\ e \end{bmatrix}, \quad A_{cl} = \begin{bmatrix} A - B_s K & B_s K \\ 0 & A - LC_T \end{bmatrix},$$

$$B_{cl}\Delta C_{cl}\bar{x} = \begin{bmatrix} I \\ I \end{bmatrix} \delta.$$

The values of B_{cl} and C_{cl} depend on how the uncertainty δ is represented in terms of Δ . For example, when $\delta = \Delta x$, we have $B_{cl} = [I \quad I]^T$ and $C_{cl} = [I \quad 0]$. Denote b and c as the smallest numbers satisfying $B_{cl}B_{cl}^T \preceq bI$ and $C_{cl}^T C_{cl} \preceq cI$. The transformations to $M - \Delta$ systems for different types of uncertainties will be studied in Section V.

A sufficient and necessary condition guaranteeing the stability of the $M - \Delta$ system is given by the following lemma.

Lemma 1 ([26], Small Gain Theorem). *The interconnected system in Fig. 1 is internally stable for all Δ with $\|\Delta\|_\infty \leq \sigma$ if and only if $\|M\|_\infty < 1/\sigma$.*

The following lemma shows a condition equivalent to the stability requirement given by Lemma 1, on time-domain matrices (A_{cl}, B_{cl}, C_{cl}) .

Lemma 2 ([27], Lemma 7.4). *Suppose $M = \left[\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & 0 \end{array} \right]$.*

Then $\|M\|_\infty < 1$ if and only if there exists a positive definite matrix $X \succ 0$ such that

$$\begin{bmatrix} C_{cl}^T \\ 0 \end{bmatrix} [C_{cl} \quad 0] + \begin{bmatrix} (A_{cl}^T X + X A_{cl}) & X B_{cl} \\ B_{cl}^T X & -I \end{bmatrix} \prec 0. \quad (10)$$

C. Submodularity Ratio

Submodularity is a diminishing-returns property of set functions, wherein the incremental benefit of adding an element to a set S decreases as more elements are added to S [28]. Let 2^Ω denote the power set of Ω . A set function $f : 2^\Omega \rightarrow R$ is *submodular* if, for any sets $S \subseteq T \subseteq \Omega$ and any element $v \in \Omega \setminus T$, $f(S \cup \{v\}) - f(S) \geq f(T \cup \{v\}) - f(T)$ [28]. Submodularity is analogous to concavity of continuous functions.

The problem of maximizing a monotone submodular function subject to a cardinality constraint can be approximately solved by polynomial-time greedy algorithms, where solutions are proved to be within a factor of $(1 - 1/e)$ of the optimal solutions [29].

The submodularity ratio measures to what extent a function f has submodular properties. The definition of submodularity ratio is given as follows.

Definition 1 (Submodularity ratio [10]). *Let $f : 2^\Omega \rightarrow \mathbb{R}$ be a non-negative set function. The submodularity ratio of f with respect to a set $U \subseteq \Omega$ and a parameter $k \geq 1$ is*

$$\gamma_{U,k} = \min_{\substack{L, S: L \cap S = \emptyset \\ L \subseteq U, |S| \leq k}} \frac{\sum_{z \in S} (f(L \cup \{z\}) - f(L))}{f(L \cup S) - f(L)}$$

For a nondecreasing function f , we have $\gamma_{U,k} \in [0, 1]$ and f is submodular if and only if $\gamma_{U,k} \geq 1$.

An important concept related to submodularity ratio is coefficient of determination, which is defined as follows.

Definition 2 (Coefficient of determination [10]). *For a linear regression problem $\min_z \|Hz - v\|_2^2$, assuming the columns of $H \in \mathbb{R}^{n \times m}$ and $v \in \mathbb{R}^n$ are normalized to have norm 1, the coefficient of determination (or R^2 statistic) is*

$$R^2 = n \bar{v}^T \bar{C}^{-1} \bar{v}$$

where $\bar{C} = H^T H/n$ and $\bar{v} = H^T v/n$.

IV. PROPOSED INPUT AND OUTPUT SELECTION FRAMEWORK

This section formulates the problem of selecting a minimal set of inputs and a minimal set of outputs that guarantee the existence of a stabilizing controller. In the first subsection, we give the exact formulation, followed by a sufficient condition in order to achieve computational feasibility. In the remaining subsections, we prove that the minimal input selection and minimal output selection to satisfy the sufficient condition are equivalent to two column-subset selection problems with bounded submodularity ratios. By exploiting the submodular structure, we present a polynomial-time greedy algorithm for input and output selections with provable optimality guarantees on the minimum number of inputs and outputs required.

A. Problem Formulation

The problem studied in this paper is to select a minimum-size set $S \subseteq \Omega_{in}$ of inputs and a minimum-size set $T \subseteq \Omega_{out}$ of outputs such that there exists a controller $u_s = -K\hat{x}$ that stabilizes the system in the presence of any uncertainty δ . In what follows, we formulate the problem mathematically and derive a sufficient condition on selections to satisfy the stability requirement.

In the equivalent $M - \Delta$ representation (8)-(9) of the system (3)-(4), the input selection S and output selection T have impact on matrix A_{cl} by determining matrices B_s and C_T respectively,

$$A_{cl} = \begin{bmatrix} A - B_s K & B_s K \\ 0 & A - LC_T \end{bmatrix}. \quad (11)$$

Assume the uncertainty block Δ has bounded singular values, i.e., $\|\Delta\|_\infty \leq \sigma$. By Lemma 1, a stable $M - \Delta$ system requires the largest singular value of the transfer function M less than $1/\sigma$. In the time domain, M has a realization in state-space system (A_{cl}, B_{cl}, C_{cl}) as defined by (8)-(9).

For any transfer function $M = C_{cl}(sI - A_{cl})^{-1}B_{cl}$, an equivalent condition to $\|M\|_\infty < 1/\sigma$ is that $\|M'\|_\infty < 1$ where $M' = \sqrt{\sigma}C_{cl}(sI - A_{cl})^{-1}\sqrt{\sigma}B_{cl}$.

By substituting M' for M in Lemma 2, we have the following result.

Corollary 1. $\|M\|_\infty < 1/\sigma$ if and only if

$$\begin{bmatrix} \sqrt{\sigma}C_{cl}^T \\ 0 \end{bmatrix} \begin{bmatrix} \sqrt{\sigma}C_{cl} & 0 \end{bmatrix} + \begin{bmatrix} A_{cl}^T X + X A_{cl} & \sqrt{\sigma}X B_{cl} \\ \sqrt{\sigma}B_{cl}^T X & -I \end{bmatrix} \prec 0 \quad (12)$$

for some $X \succ 0$.

Applying a Schur complement operation, the inequality (12) is equivalent to

$$\sigma C_{cl}^T C_{cl} + A_{cl}^T X + X A_{cl} + \sigma X B_{cl} B_{cl}^T X \prec 0. \quad (13)$$

By Lemma 1 and 2, the problem of selecting minimal sets of inputs and outputs to ensure existence of a stabilizing control robust to uncertainties, $\|\Delta\|_\infty \leq \sigma$, can be formulated as follows.

For input selection, the objective is

$$\min_{S \subseteq \Omega_{in}} |S| \quad s.t. \text{ (13) holds for some } X \succ 0. \quad (14)$$

For output selection, the objective is

$$\min_{T \subseteq \Omega_{out}} |T| \quad s.t. \text{ (13) holds for some } X \succ 0. \quad (15)$$

Both problems are subject to the system stability constraint that for some $X \succ 0$,

$$\sigma C_{cl}^T C_{cl} + A_{cl}^T X + X A_{cl} + \sigma X B_{cl} B_{cl}^T X \prec 0. \quad (16)$$

Note that (16) contains the closed-loop system matrix A_{cl} defined by Eq. (11) and hence is a function of S and T .

Without additional structure, problems (14) and (15) require exhaustive search over all selections of S and T .

In what follows, we present a sufficient condition on S and T to satisfy the stability constraint (16). The sufficient condition is established by constructing a solution X to inequality (16) and mapping the requirement on X to a constraint on the eigenvalues of A_{cl} . We show later that the sufficient constraints have bounded submodularity ratio and hence lead to computationally efficient input and output selection algorithms.

The following lemma constructs a solution X to (16).

Lemma 3. Suppose that $C_{cl}^T C_{cl} \preceq cI$ and $B_{cl} B_{cl}^T \preceq bI$. For any Hurwitz matrix A_{cl} , suppose that there exists $\epsilon > 0$ such that $X \prec \epsilon I$ where X is defined by

$$X = \int_0^\infty e^{A_{cl}^T t} (\sigma c I + \sigma b \epsilon^2 I) e^{A_{cl} t} dt. \quad (17)$$

Then X is a solution to (16).

Proof. Suppose $C_{cl}^T C_{cl} \preceq cI$, $B_{cl} B_{cl}^T \preceq bI$. Then the expression in (16) is bounded by

$$\begin{aligned} \sigma C_{cl}^T C_{cl} + A_{cl}^T X + X A_{cl} + \sigma X B_{cl} B_{cl}^T X \\ \preceq \sigma c I + A_{cl}^T X + X A_{cl} + \sigma b X^2. \end{aligned}$$

Thus, any positive definite matrix X that satisfies

$$\sigma c I + A_{cl}^T X + X A_{cl} + \sigma b X^2 \prec 0 \quad (18)$$

is a solution to inequality (16).

In what follows, we prove that the X constructed in (17) is a solution to (18).

When $X \prec \epsilon I$, it can be shown that $X^2 \prec \epsilon^2 I$ (see appendix for proof). Substituting (17) into (18) gives

$$\begin{aligned} \sigma c I + A_{cl}^T X + X A_{cl} + \sigma b X^2 \\ = \sigma c I + \sigma b X^2 + \int_0^\infty A_{cl}^T e^{A_{cl}^T t} (\sigma c I + \sigma b \epsilon^2 I) e^{A_{cl} t} dt \\ + \int_0^\infty e^{A_{cl}^T t} (\sigma c I + \sigma b \epsilon^2 I) e^{A_{cl} t} A_{cl} dt \\ = \sigma c I + \sigma b X^2 + \int_0^\infty \frac{d}{dt} \left(e^{A_{cl}^T t} (\sigma c I + \sigma b \epsilon^2 I) e^{A_{cl} t} \right) dt \\ = \sigma c I + \sigma b X^2 + e^{A_{cl}^T t} (\sigma c I + \sigma b \epsilon^2 I) e^{A_{cl} t} \Big|_{t=0}^\infty \\ = \sigma c I + \sigma b X^2 - (\sigma c I + \sigma b \epsilon^2 I) \\ = \sigma b (X^2 - \epsilon^2 I) \prec 0, \end{aligned}$$

which implies that the X constructed in (17) is a solution to inequality (18) and hence is a solution to inequality (16). \square

By Lemma 3, in order to meet the system stability constraint (16), it is sufficient to have the matrix A_{cl} satisfy the following: for some $\epsilon > 0$,

$$\int_0^\infty e^{A_{cl}^T t} (\sigma c I + \sigma b \epsilon^2 I) e^{A_{cl} t} dt \prec \epsilon I. \quad (19)$$

The following lemma simplifies the inequality (19), which results in a sufficient condition to constraint (16).

Lemma 4. Let λ_0 denote the eigenvalue of A_{cl} with the largest real part. Define $\alpha = \lambda_1(A_{cl} + A_{cl}^T)/(2\text{Re}(\lambda_0))$, where the notation $\lambda_1(\cdot)$ refers to the largest eigenvalue. Assume A_{cl} is asymptotically stable, i.e., $\text{Re}(\lambda_0) < 0$. Then the inequality (19) holds if

$$\text{Re}(\lambda_0) < -\sigma\sqrt{bc}/\alpha. \quad (20)$$

Proof. Given the fact that a symmetric matrix $(\epsilon I - X)$ is positive definite if and only if all its eigenvalues are positive, the inequality $X \prec \epsilon I$ in (19) holds if the largest eigenvalue of X is smaller than ϵ , i.e., $\lambda_1(X) < \epsilon$. The constructed X in (19) is a solution of the Lyapunov equation

$$A_{cl}^T X + X A_{cl} + Q = 0, \quad \text{where } Q = (\sigma c + \sigma b \epsilon^2) I.$$

The eigenvalues of any solution X are bounded by $\lambda_1(X) \leq -\lambda_1(Q)/\lambda_1(A_{cl} + A_{cl}^T)$ when $A_{cl} + A_{cl}^T \prec 0$ [30]. Substituting $\lambda_1(A_{cl} + A_{cl}^T)$ with $2\alpha\text{Re}(\lambda_0)$, we have

$$\lambda_1(X) \leq \frac{\sigma c + \sigma b \epsilon^2}{-2\alpha\text{Re}(\lambda_0)}.$$

In order to guarantee $\lambda_1(X) < \epsilon$, it suffices to require

$$\operatorname{Re}(\lambda_0) < -\frac{\sigma}{2\alpha}\left(\frac{c}{\epsilon} + b\epsilon\right). \quad (21)$$

After finding the maximum value of the right-hand-side function in (21) respect to ϵ , we have the inequality (21) holds for some $\epsilon > 0$ if $\operatorname{Re}(\lambda_0) < -\sigma\sqrt{bc}/\alpha$. \square

By Lemma 3 and 4, the constraint (16) for input and output selections is relaxed to a sufficient condition (20) on eigenvalues of A_{cl} , which still guarantees the design of a feedback control K that can stabilize the system in the presence of uncertainty Δ .

Given that A_{cl} in (11) is a block upper triangular matrix with matrices $A - B_s K$ and $A - LC_T$ on the diagonal, the eigenvalue condition (20) is satisfied if both $A - B_s K$ and $A - LC_T$ have eigenvalues less than $-\sigma\sqrt{bc}/\alpha$. Thus, the sufficient condition guaranteeing system stability becomes

$$\operatorname{Re}(\lambda(A - B_s K)) < -\sigma\sqrt{bc}/\alpha \quad (22)$$

for input selection S , and

$$\operatorname{Re}(\lambda(A - LC_T)) < -\sigma\sqrt{bc}/\alpha \quad (23)$$

for output selection T , where the notation $\lambda(\cdot)$ refers to all eigenvalues of a given matrix.

Hence, the input selection problem can be formulated as

$$\begin{aligned} \min_{S \subseteq \Omega_{in}} & |S| \\ \text{s.t.} & \operatorname{Re}(\lambda(A - B_s K)) < -\sigma\sqrt{bc}/\alpha \end{aligned} \quad (24)$$

and the output selection problem can be formulated as

$$\begin{aligned} \min_{T \subseteq \Omega_{out}} & |T| \\ \text{s.t.} & \operatorname{Re}(\lambda(A - LC_T)) < -\sigma\sqrt{bc}/\alpha \end{aligned} \quad (25)$$

where $C_{cl}^T C_{cl} \preceq cI$, $B_{cl} B_{cl}^T \preceq bI$.

The input selection is to achieve the stabilizability of the system while the output selection is to achieve the system detectability, both in the presence of uncertainties. For a nominal system, i.e., $\Delta = 0$, our stability threshold ($-\sigma\sqrt{bc}/\alpha = 0$) is consistent with the existing literature [18].

By the fact that $2\operatorname{Re}(\lambda_0) \leq \lambda_1(A_{cl} + A_{cl}^T) < 0$ [31], we have that α is in the range $(0, 1]$. We take the following approach to approximating the solution to (24) and (25). We first set $\alpha = 1$, yielding the constraints

$$\operatorname{Re}(\lambda(A - B_s K)) < -\sigma\sqrt{bc}, \quad (26)$$

$$\operatorname{Re}(\lambda(A - LC_T)) < -\sigma\sqrt{bc}, \quad (27)$$

and select a set of inputs and outputs using the procedure enumerating in the following sections. After synthesizing a controller and computing the matrices A_{cl} , B_{cl} , and C_{cl} , if the condition of Lemma 2 does not hold, then we reduce the value of α and reuse the procedure of the following sections, but with the constraints $\operatorname{Re}(\lambda(A - B_s K)) < -\sigma\sqrt{bc}/\alpha$ and $\operatorname{Re}(\lambda(A - LC_T)) < -\sigma\sqrt{bc}/\alpha$. We continue this procedure until an input and an output sets satisfying Lemma 2 are selected. In practice, during the simulations described in Section VI, the conditions (26) and (27) were sufficient to

ensure system stability during all simulation trials. This is because the small gain theorem (Lemma 1) is known to be conservative and hence by picking $\alpha = 1$, the conditions (26) and (27) are in general sufficient to ensure stability.

B. Submodularity Ratio of Input Selection Constraint

In this section, we establish a metric for input selection (24) and prove the metric has a bounded submodularity ratio.

First, we present a lemma, as follows, which shows a sufficient condition that guarantees the existence of a feedback control K satisfying the input selection constraint (24), i.e., $\operatorname{Re}(\lambda(A - B_s K)) < -\sigma\sqrt{bc}$.

Lemma 5. *Let*

$$\dot{x}(t) = Ax(t) + B_s u(t) \quad (28)$$

$$y(t) = Cx(t) \quad (29)$$

be a fully observable control system. If all of the eigenvectors v_i of A with eigenvalues λ_i satisfying $\operatorname{Re}(\lambda_i) \geq \lambda_1$ lie in the span of the controllability matrix of the system, then there exists a feedback control K such that all eigenvalues of the closed-loop system $A - B_s K$ satisfy $\operatorname{Re}(\lambda(A - B_s K)) < \lambda_1$.

Proof. The proof can be found in appendix, which is a straightforward generalization of Theorem 8.3 of [18] and is included for completeness. \square

By Lemma 5, it suffices to ensure the existence of a feedback control K that satisfies constraint (24), if all eigenvectors v_i of A corresponding to “undesired modes” ($\operatorname{Re}(\lambda_i) \geq -\sigma\sqrt{bc}$) lie in the span of the controllability matrix of the system (A, B_s) , i.e.,

$$\{v_i : \operatorname{Re}(\lambda_i) \geq -\sigma\sqrt{bc}\} \in \operatorname{span}(\mathcal{C}(S)), \quad (30)$$

where $\mathcal{C}(S) = [B_s \ AB_s \ A^2 B_s \ \dots \ A^{n-1} B_s]$ denotes the controllability matrix of system (A, B_s) .

These “undesired modes” with eigenvalues satisfying $\operatorname{Re}(\lambda_i) \geq -\sigma\sqrt{bc}$ refer to dynamics in the system (3)-(4) that are unstable in the presence of uncertainty Δ with $\|\Delta\|_\infty \leq \sigma$.

It is shown that for a linear system as defined in (28), the span of its controllability matrix equals to the span of its controllability Gramian [32], i.e.,

$$\operatorname{span}(\mathcal{C}(S)) = \operatorname{span}(W(S)), \quad (31)$$

where $W(S)$ is the controllability Gramian defined by

$$W(S) = \int_{t_0}^{t_1} e^{A(t-t_0)} B_s B_s^T e^{A^T(t-t_0)} dt \quad (32)$$

for some $t_1 > t_0$.

By (30) and (31), we construct a metric for the input selection in the presence of bounded uncertainty $\|\Delta\|_\infty \leq \sigma$, given by

$$f(S) \triangleq \sum_{i: \operatorname{Re}(\lambda_i) \geq -\sigma\sqrt{bc}} \operatorname{dist}^2(v_i, \operatorname{span}(W(S))), \quad (33)$$

where $\operatorname{dist}(\cdot)$ denotes the Euclidean distance; λ_i and v_i are eigenvalues and corresponding eigenvectors of A , respectively; and b, c are constants of the assumption $C_{cl}^T C_{cl} \preceq cI$, $B_{cl} B_{cl}^T \preceq bI$.

Given any input selection S , the function $f(S)$ measures the distance between the span of the controllability Gramian and the eigenvectors of “undesired modes” where eigenvalues $\text{Re}(\lambda_i) \geq -\sigma\sqrt{bc}$. Intuitively, this metric is a measure of how close the unstable modes of the system (3)-(4) are to being controllable.

To ensure the existence of a feedback control K that satisfies the input selection constraint (24), it suffices to require

$$f(S) = 0. \quad (34)$$

In the following, we prove that the function $f(S)$ has bounded submodularity ratio. Before giving the bound, we first define some notations as follows.

Let $C = [B \ AB \ A^2B \ \dots \ A^{n-1}B]$ be the controllability matrix and let $P \in \mathbb{R}^{np \times np}$ be a nonsingular matrix that normalizes C to $C' = CP$ where each column of C' has norm 1. For simplicity, we use C_s to denote the function $C(S)$ in the following and denote $C'_s = C_s P$.

Define $\bar{C} \in \mathbb{R}^{np \times np} = C'^T C' / n$ and $\bar{C}_s = C_s'^T C'_s / n$. We note that \bar{C}_s is a submatrix¹ of \bar{C} with rows and columns selected by set S . For any matrix \bar{C}_s , we denote its smallest eigenvalue as $\lambda_{\min}(\bar{C}_s)$ and let $\lambda_{\min}(\bar{C}, k) = \min_{S: |S|=k} \lambda_{\min}(\bar{C}_s)$.

The submodularity ratio of the input selection constraint (34) is described by the following theorem.

Theorem 1. *Let $\gamma_{U,k}$ be the submodularity ratio of the set function $f(S)$ defined in (33). Then for any set $U \subseteq \Omega_{in}$ and $k \geq 1$, the submodularity ratio $\gamma_{U,k}$ is bounded by*

$$\gamma_{U,k} \geq \lambda_{\min}(\bar{C}, k + |U|) \geq \lambda_{\min}(\bar{C}).$$

Before giving the proof, we first present some preliminary results as follows.

Given any vector v with $\|v\|_2 = 1$, consider the function

$$f_v(S) = \text{dist}^2(v, \text{span}(C(S))) = \min_z \|C(S)z - v\|_2^2.$$

Lemma 6. *For any nonsingular matrix P , the following equality holds:*

$$\min_z \|C_s z - v\|_2^2 = \min_z \|C_s P z - v\|_2^2.$$

Proof. Let $z'^* = \arg \min_z \|C_s P z - v\|_2^2$. Then $z'^* = P^{-1}(C_s^T C_s)^{-1} P^{-T} P^T C_s^T v = P^{-1}(C_s^T C_s)^{-1} C_s^T v = P^{-1} z^*$, where $z^* = \arg \min_z \|C_s z - v\|_2^2$. Thus,

$$\begin{aligned} \min_z \|C_s P z - v\|_2^2 &= \|C_s P z'^* - v\|_2^2 = \|C_s P P^{-1} z^* - v\|_2^2 \\ &= \|C_s z^* - v\|_2^2 = \min_z \|C_s z - v\|_2^2, \end{aligned}$$

completing the proof. \square

By Lemma 6, the function $f(S)$ is equivalent to

$$f_v(S) = \min_z \|C'_s z - v\|_2^2. \quad (35)$$

Denote z_s^* as the optimal vector that solves $\min_z \|C'_s z - v\|_2^2$. The value of z_s^* is given by $z_s^* = (C_s'^T C'_s)^{-1} C_s'^T v$ [33]. Knowing that $v - C'_s z_s^*$ and $C'_s z_s^*$ are orthogonal, we have

$$\begin{aligned} f_v(S) &= \|C'_s z_s^* - v\|_2^2 = \|v\|_2^2 - \|C'_s z_s^*\|_2^2 \\ &= 1 - \|C'_s z_s^*\|_2^2. \end{aligned}$$

¹ \bar{C}_s consists of $n \times |S|$ rows and columns selected from \bar{C} .

Define a new set function

$$g_v(S) = \|C'_s z_s^*\|_2^2 = \|P_s v\|_2^2, \quad (36)$$

where $P_s = C_s'(C_s'^T C'_s)^{-1} C_s'^T$ is the projection matrix for orthogonal projection onto the span of columns of C'_s , and hence

$$g_v(S) = 1 - f_v(S). \quad (37)$$

Lemma 7 ([34]). *The function $g_v(S)$ is the R^2 statistic (Definition 2) for the linear regression problem (35).*

Proof. By definition,

$$\begin{aligned} g_v(S) &= \|C'_s (C_s'^T C'_s)^{-1} C_s'^T v\|_2^2 \\ &= (C_s'^T v)^T (C_s'^T C'_s)^{-1} (C_s'^T v) = n \bar{v}_s^T \bar{C}_s^{-1} \bar{v}_s, \end{aligned}$$

where $\bar{C}_s = C_s'^T C'_s / n$ and $\bar{v}_s = C_s'^T v / n$, satisfying the definition of R^2 statistic for the linear regression (35) [10]. \square

Lemma 8 ([10]). *Let $\gamma'_{U,k}$ be the submodularity ratio of the R^2 statistic (36). Then*

$$\gamma'_{U,k} \geq \lambda_{\min}(\bar{C}, k + |U|) \geq \lambda_{\min}(\bar{C}). \quad (38)$$

By Lemma 7 and 8, the function $g_v(S)$ has submodularity ratio $\gamma'_{U,k}$ bounded by (38). Using this result, now we are ready to give the proof of Theorem 1.

Proof of Theorem 1: We first show that $f_v(S)$ and $g_v(S)$ have the same submodularity ratio $\gamma'_{U,k}$. From (37), we have $g_v(S) = 1 - f_v(S)$. By definition,

$$\begin{aligned} \gamma'_{U,k} &= \min_{\substack{L, S: L \cap S = \emptyset \\ L \subseteq U, |S| \leq k}} \frac{\sum_{z \in S} (g_v(L \cup \{z\}) - g_v(L))}{g_v(L \cup S) - g_v(L)} \\ &= \min_{\substack{L, S: L \cap S = \emptyset \\ L \subseteq U, |S| \leq k}} \frac{\sum_{z \in S} ((1 - f_v(L \cup \{z\})) - (1 - f_v(L)))}{(1 - f_v(L \cup S)) - (1 - f_v(L))} \\ &= \min_{\substack{L, S: L \cap S = \emptyset \\ L \subseteq U, |S| \leq k}} \frac{\sum_{z \in S} (f_v(L \cup \{z\}) - f_v(L))}{f_v(L \cup S) - f_v(L)} \end{aligned}$$

Then we show that $f(S)$ has submodularity ratio $\gamma_{U,k}$ bounded by $\gamma'_{U,k}$. By Eq. (31) and Lemma 6, we have $f(S) = \sum_i f_{v_i}(S)$ where v_i is the eigenvector of the i th “undesired mode” with eigenvalue $(\text{Re}(\lambda) \geq -\sigma\sqrt{bc})$. Then

$$\begin{aligned} \gamma_{U,k} &= \min_{\substack{L, S: L \cap S = \emptyset \\ L \subseteq U, |S| \leq k}} \frac{\sum_{z \in S} (f(L \cup \{z\}) - f(L))}{f(L \cup S) - f(L)} \\ &= \min_{\substack{L, S: L \cap S = \emptyset \\ L \subseteq U, |S| \leq k}} \frac{\sum_i \sum_{z \in S} (f_{v_i}(L \cup \{z\}) - f_{v_i}(L))}{\sum_i (f_{v_i}(L \cup S) - f_{v_i}(L))} \\ &\geq \min_{\substack{L, S: L \cap S = \emptyset \\ L \subseteq U, |S| \leq k}} \left(\min_i \frac{\sum_{z \in S} (f_{v_i}(L \cup \{z\}) - f_{v_i}(L))}{(f_{v_i}(L \cup S) - f_{v_i}(L))} \right) \\ &\geq \gamma'_{U,k} \end{aligned}$$

By Lemma 8, we have $\gamma_{U,k} \geq \gamma'_{U,k} \geq \lambda_{\min}(\bar{C}, k + |U|) \geq \lambda_{\min}(\bar{C})$, completing the proof. \square

C. Submodularity Ratio of Output Selection Constraint

In this section, we derive a metric for output selection (25) that has bounded submodularity ratio by analogy with input selection.

By condition (25), to achieve a detectable system in the presence of uncertainty Δ with $\|\Delta\|_\infty \leq \sigma$ requires that

$$\operatorname{Re}(\lambda(A - LC_T)) < -\sigma\sqrt{bc},$$

or equivalently,

$$\operatorname{Re}(\lambda(A^T - C_T^T L^T)) < -\sigma\sqrt{bc}, \quad (39)$$

where A^T, C_T^T, L^T are the transpose of matrices A, C_T, L respectively.

By duality theory, Lemma 5 implies that for the system (A^T, C_T^T) , if all of the eigenvectors v'_i of A^T with eigenvalues λ_i satisfying $\lambda_i \geq \lambda_0$ lie in the span of the matrix

$$M(T) = \int_{t_0}^{t_1} e^{A^T(t-t_0)} C_T^T C_T e^{A(t-t_0)} dt, \quad (40)$$

there exists a matrix L such that all eigenvalues of the closed-loop system $A^T - C_T^T L^T$ satisfy $\operatorname{Re}(\lambda(A^T - C_T^T L^T)) < \lambda_0$.

It is shown that the matrix $M(T)$ defined by (40) is the observability Gramian of the system (A, C) . Thus, a sufficient condition to ensure system detectability in the presence of uncertainty $\|\Delta\|_\infty \leq \sigma$ is that all eigenvectors v'_i of A^T with eigenvalues $\lambda_i \geq -\sigma\sqrt{bc}$ lie in the span of the observability Gramian $M(S)$ of the system (A, C) , i.e.,

$$\{v'_i : \operatorname{Re}(\lambda_i) \geq -\sigma\sqrt{bc}\} \in \operatorname{span}(M(T)). \quad (41)$$

We present a metric for output selection as follows, which measures how far the system is from being detectable in the presence of bounded uncertainty $\|\Delta\|_\infty \leq \sigma$,

$$g(T) \triangleq \sum_{i: \operatorname{Re}(\lambda_i) \geq -\sigma\sqrt{bc}} \operatorname{dist}^2(v'_i, \operatorname{span}(M(T))), \quad (42)$$

where v'_i are eigenvectors of the matrix A^T .

For output selection, we have the main results, as follows, similar to the input selection. To ensure the existence of a matrix L that satisfies the output selection constraint (25), it suffices to require

$$g(T) = 0. \quad (43)$$

The submodularity ratio of the constraint (43) is described by the following theorem.

Theorem 2. *Let $\eta_{U,k}$ be the submodularity ratio of the set function $g(T)$ defined in (42). Then for any set $U \subseteq \Omega$ and $k \geq 1$, the submodularity ratio $\eta_{U,k}$ is bounded by*

$$\eta_{U,k} \geq \lambda_{\min}(\bar{O}, k + |U|) \geq \lambda_{\min}(\bar{O}).$$

where \bar{O} and $\lambda_{\min}(\cdot)$ are defined as follows. Let

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}^T \quad \text{and} \quad \mathcal{O}(T) = \begin{bmatrix} C_T \\ C_T A \\ C_T A^2 \\ \vdots \\ C_T A^{n-1} \end{bmatrix}^T$$

Let $D \in \mathbb{R}^{nm \times nm}$ be a nonsingular matrix that normalizes the observability matrix \mathcal{O} to $\mathcal{O}' = \mathcal{O}D$ where each column of \mathcal{O}' has norm 1, and hence $\mathcal{O}'(T) = \mathcal{O}(T)D$. We define $\bar{O} \in \mathbb{R}^{nm \times nm} = \mathcal{O}'^T \mathcal{O}' / n$ and $\bar{O}(T) = \mathcal{O}'(T)^T \mathcal{O}'(T) / n$. $\lambda_{\min}(\bar{O}(T))$ refers to the smallest eigenvalue of $\bar{O}(T)$ and $\lambda_{\min}(\bar{O}, k) = \min_{T: |T|=k} \lambda_{\min}(\bar{O}(T))$.

Proof. The proof can be established directly by following the same steps as the proof of Theorem 1, while the controllability Gramian W and the controllability matrix C are replaced by the observability Gramian M and the observability matrix (transpose) \mathcal{O} . \square

D. Minimal Input and Output Selection Algorithm

Using the metrics we established in previous sections, the problem of selecting the minimal set of input nodes to ensure system stabilizability in the presence of uncertainties can be formulated as

$$\begin{aligned} \min_{S \subseteq \Omega_{in}} |S| \\ \text{s.t. } f(S) = 0 \end{aligned} \quad (44)$$

and the output selection for system detectability is given as

$$\begin{aligned} \min_{T \subseteq \Omega_{out}} |T| \\ \text{s.t. } g(T) = 0 \end{aligned} \quad (45)$$

For some control systems, the input selection and output selection are dependent to each other. For example, the wide-area damping system in a power grid selects generators as input to participate in the centralized control while observing rotor speeds at these selected generators as system outputs [35]. For such systems that input and output selections share the same ground set denoted by Ω_{joint} , we propose a joint selection formulation

$$\begin{aligned} \min_{S \subseteq \Omega_{joint}} |S| \\ \text{s.t. } f(S) + g(S) = 0 \end{aligned} \quad (46)$$

Note that $f + g$ in constraint (46) is the sum of two monotone decreasing set functions with submodularity ratios $\gamma_{U,k}$ and $\eta_{U,k}$, respectively. By definition, the submodularity ratio of $f + g$ is bounded by $\min(\gamma_{U,k}, \eta_{U,k})$. Since $\gamma_{U,k} \geq \lambda_{\min}(\bar{C}, k + |U|)$ and $\eta_{U,k} \geq \lambda_{\min}(\bar{O}, k + |U|)$, thus a lower bound of the submodularity ratio of $f + g$ is $\min(\lambda_{\min}(\bar{C}, k + |U|), \lambda_{\min}(\bar{O}, k + |U|))$. We also note that, since $f(S) \geq 0$ and $g(S) \geq 0$, the constraint (46) is satisfied if and only if $f(S) = 0$ and $g(S) = 0$.

The sufficient conditions (44)-(45) for selecting input and output nodes guarantee the existence of a stabilizing controller that can drive the system state to the origin in the presence of uncertainties. With selected inputs and outputs satisfying these conditions, one can find such a stabilizing controller using robust control design with pole placement constraints, such as the LMI-based H_∞ synthesis proposed in [36].

We now present a polynomial-time algorithm, shown as Algorithm 1, that approximately solves the above input selection, output selection, and joint selection problems, with an optimality bound on the solution set cardinality.

Algorithm 1 Algorithm for selecting minimum-size set of inputs/outputs for a stabilizing controller.

```

1: procedure MINSET( $\Omega, F$ )
2:   Input: Ground set  $\Omega$  of inputs/outputs, i.e.,  $\Omega = \Omega_{in}$ 
   for input selection,  $\Omega = \Omega_{out}$  for output selection, or  $\Omega = \Omega_{joint}$ 
   for joint selection;
   Supermodular metric  $F$  in constraint, i.e.,  $F = f$  for
   input selection,  $F = g$  for output selection, or  $F = f + g$ 
   for joint selection.
3:   Output: Set of inputs/outputs  $S$  to participate in the
   system control.
4:   Initialization:  $S \leftarrow \emptyset$ 
5:   while  $F(S) > 0$  and  $S \neq \Omega$  do
6:     for  $v \in \Omega \setminus S$  do
7:        $\delta_v \leftarrow F(S) - F(S \cup \{v\})$ 
8:     end for
9:      $v^* \leftarrow \arg \max_v \delta_v$ 
10:     $S \leftarrow S \cup \{v^*\}$ 
11:  end while
12:  return  $S$ 
13: end procedure

```

For the input selection problem (44), we set $\Omega = \Omega_{in}$ and $F = f$. For the output selection problem (45), we set $\Omega = \Omega_{out}$ and $F = g$. For the joint selection problem (46), we set $\Omega = \Omega_{joint}$ and $F = f + g$.

The algorithm proceeds as follows. Given Ω and F , a set of selected inputs/outputs, S , is initialized to be empty. At each iteration, the algorithm searches an element v from Ω/S that maximizes $F(S) - F(S \cup \{v\})$ and adds it to the selection set S . The algorithm terminates when $F(S)$ is reduced to 0 or $S = \Omega$.

The input and output sets selected by Algorithm 1 have cardinalities within a factor of the true optimal solution of the original problem (44), (45), or (46). The optimality bound of Algorithm 1 is defined by the following lemma.

Proposition 1. *Let S^* denote the true optimal solution to the input/output selection problem (44), (45), or (46) that has the form $\{\min_{S \in \Omega} |S| : F(S) = 0\}$ and let S denote the solution returned by Algorithm 1. Then we have*

$$\frac{|S| - 1}{|S^*|} \leq \frac{1}{\gamma_0} \log \frac{F(\emptyset)}{F(S_{t-1})},$$

where S_{t-1} denotes the selected input set S at the second-to-last iteration of Algorithm 1 and γ_0 is the lower bound of the submodularity ratio of F .

For input selection (44), $\gamma_0 = \lambda_{\min}(\bar{C}, 2|S|)$; For output selection (45), $\gamma_0 = \lambda_{\min}(\bar{O}, 2|S|)$; For joint selection (46), $\gamma_0 = \min(\lambda_{\min}(\bar{C}, 2|S|), \lambda_{\min}(\bar{O}, 2|S|))$.

Proof. Suppose F has submodularity ratio $\gamma_{U,k}$ for any given set U and integer k . Let $\{s_1, \dots, s_k\}$ denote the elements selected by the Algorithm 1 in the first k iterations. By the definition of submodularity ratio, we have

$$\frac{\sum_{z \in S^*} (F(\emptyset) - F(\{z\}))}{F(\emptyset) - F(S^*)} \geq \gamma_{S,|S|} \geq \gamma_0 \quad (47)$$

The lower bound γ_0 depends on function F . When $F = f$, by Theorem 1, we have $\gamma_{S,|S|} \geq \lambda_{\min}(\bar{C}, 2|S|)$. When $F = g$, by Theorem 2, we have $\gamma_{S,|S|} \geq \lambda_{\min}(\bar{O}, 2|S|)$. When $F = f + g$, we have $\gamma_{S,|S|} \geq \min(\lambda_{\min}(\bar{C}, 2|S|), \lambda_{\min}(\bar{O}, 2|S|))$.

Inequality (47) implies that

$$|S^*| (F(\emptyset) - F(\{s_1\})) \geq \gamma_0 (F(\emptyset) - F(S^*)),$$

or equivalently,

$$F(\{s_1\}) - F(S^*) \leq \left(1 - \frac{\gamma_0}{|S^*|}\right) (F(\emptyset) - F(S^*)).$$

Similarly, we have that

$$\frac{\sum_{z \in S^*} (F(s_1) - F(\{z, s_1\}))}{F(\{s_1\}) - F(S^* \cup \{s_1\})} \geq \gamma_0$$

implying that

$$\begin{aligned} F(\{s_1\}) - F(\{s_1, s_2\}) &\geq \frac{\gamma_0}{|S^*|} (F(\{s_1\}) - F(S^* \cup \{s_1\})) \\ &\geq \frac{\gamma_0}{|S^*|} (F(\{s_1\}) - F(S^*)), \end{aligned}$$

which is further equivalent to

$$\begin{aligned} F(\{s_1, s_2\}) - F(S^*) &\leq \left(1 - \frac{\gamma_0}{|S^*|}\right) (F(\{s_1\}) - F(S^*)) \\ &\leq \left(1 - \frac{\gamma_0}{|S^*|}\right)^2 (F(\{\emptyset\}) - F(S^*)). \end{aligned}$$

By mathematical induction, we have that

$$F(\{s_1, \dots, s_k\}) - F(S^*) \leq \left(1 - \frac{\gamma_0}{|S^*|}\right)^k (F(\{\emptyset\}) - F(S^*)).$$

Note that $F(S^*) = 0$ and hence

$$F(\{s_1, \dots, s_k\}) \leq \left(1 - \frac{\gamma_0}{|S^*|}\right)^k F(\{\emptyset\}).$$

Use the fact that $1 - 1/z \leq \log(z), \forall z \geq 1$. Then when

$$k = \frac{|S^*|}{\gamma_0} \log \frac{F(\emptyset)}{F(S_{t-1})},$$

we have

$$F(\{s_1, \dots, s_k\}) \leq F(S_{t-1}),$$

implying that one additional element is sufficient. Hence

$$\frac{|S| - 1}{|S^*|} \leq \frac{1}{\gamma_0} \log \frac{F(\emptyset)}{F(S_{t-1})}. \quad \square$$

For a set $\Omega = \{1, \dots, |\Omega|\}$, Algorithm 1 terminates in at most $|\Omega|$ iterations. For a system with n states, each iteration solves at most n least-squares problems where each problem requires computation of $O(n^3)$. Thus, the overall computational complexity of Algorithm 1 is $O(|\Omega|n^4)$.

We note that the optimality bound in Proposition 1 can be arbitrarily small for certain system matrices. This is consistent with the claim that minimal reachability is hard to approximate as presented in [21].

V. ANALYSIS FOR DIFFERENT UNCERTAINTIES

In this section, we analyze our proposed input and output selection approach for different uncertainty cases. For additive and multiplicative uncertainties in system matrix A , we show a direct transformation to the $M - \Delta$ loop model and give the specific values of the thresholds in metrics f and g . For uncertain time delay in output y , we first incorporate the delayed dynamics into the original system and then transform the modified linear system to the $M - \Delta$ model.

We also study the structured uncertainty in matrix A . From a known robustness criterion, we derive a sufficient condition on eigenvalues of the closed-loop system matrix A_{cl} , such that metrics similar to (33) and (42) that have bounded submodularity ratios can be established for input and output selection problems.

A. Additive Uncertainty in Matrix A

A linear system with additive perturbation Δ in matrix A has dynamics

$$\begin{aligned}\dot{x}(t) &= (A + \Delta)x(t) + Bu(t) \\ y(t) &= Cx(t).\end{aligned}$$

When matching this system to our model (1)-(2), the uncertain term $\delta(t) = \Delta x(t)$.

In the equivalent $M - \Delta$ loop representation (8)-(9), we have $B_{cl} = [I \ I]^T$ and $C_{cl} = [I \ 0]$.

By (20), we have the stability condition $\text{Re}(\lambda(A_{cl})) < -\sigma\sqrt{bc}$, assuming $B_{cl}B_{cl}^T \preceq bI$, $C_{cl}^T C_{cl} \preceq cI$ and $\|\Delta\|_\infty \leq \sigma$. In this case, we get $b = 2$ and $c = 1$ by the fact that

$$B_{cl}B_{cl}^T = \begin{bmatrix} I & I \\ I & I \end{bmatrix} \preceq 2 \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \quad C_{cl}^T C_{cl} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \preceq \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Therefore, for additive uncertainties in matrix A , the metric $f(S)$ defined by (33) for input selection guaranteeing system stabilizability becomes

$$f(S) = \sum_{i: \text{Re}(\lambda_i) \geq -\sigma\sqrt{2}} \text{dist}^2(v_i, \text{span}(W(S))), \quad (48)$$

and the metric $g(T)$ for output selection guaranteeing system detectability is given by

$$g(T) = \sum_{i: \text{Re}(\lambda_i) \geq -\sigma\sqrt{2}} \text{dist}^2(v'_i, \text{span}(M(T))). \quad (49)$$

These changes in metrics f and g therefore impact the Algorithm 1 by specifying the parameters b and c in the eigenvalue threshold $(-\sigma\sqrt{bc})$.

B. Multiplicative Uncertainty in Matrix A

For a multiplicative perturbation in matrix A , it changes the system dynamics to

$$\begin{aligned}\dot{x}(t) &= (I + \Delta)Ax(t) + Bu(t) \\ y(t) &= Cx(t),\end{aligned}$$

where Δ is assumed to be bounded by $\|\Delta\|_\infty \leq \sigma$. In terms of our model (1)-(2), the uncertain term $\delta(t) = \Delta Ax(t)$.

In this case, the equivalent $M - \Delta$ system has matrices $B_{cl} = [I \ I]^T$ and $C_{cl} = [A \ 0]$. By the same fact that $B_{cl}B_{cl}^T \preceq 2I$ as shown in previous case, we have $b = 2$.

Let ρ denote the largest singular value of A , i.e., $\|A\|_\infty = \rho$. Then the following inequality holds,

$$C_{cl}^T C_{cl} = \begin{bmatrix} A^T A & 0 \\ 0 & 0 \end{bmatrix} \preceq \rho^2 \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

which implies that $c = \rho^2$.

Hence, the metric $f(S)$ for input selection in the presence of a multiplicative uncertainty in matrix A is given by

$$f(S) = \sum_{i: \text{Re}(\lambda_i) \geq -\sigma\rho\sqrt{2}} \text{dist}^2(v_i, \text{span}(W(S))), \quad (50)$$

and the output selection metric $g(T)$ becomes

$$g(T) = \sum_{i: \text{Re}(\lambda_i) \geq -\sigma\rho\sqrt{2}} \text{dist}^2(v'_i, \text{span}(M(T))). \quad (51)$$

In this case, the Algorithm 1 selects inputs/outputs based on a different threshold $(-\sigma\rho\sqrt{2})$ that relays on the largest singular value of A compared to the additive uncertainty case.

C. Uncertain Time Delay in Output y

For a control system with an uncertain time delay τ in output y , the delayed output is denoted by $y_d(t) = y(t - \tau)$. In frequency domain, the output delay can be represented by a first-order Padé approximation [37],

$$y_d = e^{-s\tau} y \approx \frac{-0.5s\tau + 1}{0.5s\tau + 1} y. \quad (52)$$

A state-space realization of the transfer function (52) is given by

$$\dot{x}_d(t) = \frac{2}{\tau} (-x_d(t) + 2y(t)) \quad (53)$$

$$y_d(t) = x_d(t) - y(t). \quad (54)$$

Consider a system with m outputs $\{y_1, \dots, y_m\}$. When each output y_i has a different delay τ_i , we define a diagonal matrix of delays by $\Gamma = \text{diag}(\tau_1, \dots, \tau_m)$. By involving m such delayed dynamics (53)-(54) into the original system ($\dot{x} = Ax + Bu$, $y = Cx$), the overall system dynamics are given by

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_d(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 4\Gamma^{-1}C & -2\Gamma^{-1} \end{bmatrix} \begin{bmatrix} x(t) \\ x_d(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) \quad (55)$$

$$y_d(t) = [-C \ I] \begin{bmatrix} x(t) \\ x_d(t) \end{bmatrix}, \quad (56)$$

where $x_d \in \mathbb{R}^m$ is the state vector of delayed dynamics; $y_d \in \mathbb{R}^m$ is the vector of delayed outputs.

Suppose that each time delay τ_i is varying within a certain known interval such that $\Gamma^{-1} = \Gamma_0^{-1} + \Delta$, where Γ_0^{-1} is the nominal values of delays and Δ is the uncertain part of delays bounded by $\|\Delta\|_\infty \leq \sigma$. By defining a new state vector $\tilde{x} = [x, x_d]^T$, then the system (55)-(56) can be mapped to our system model (1)-(2),

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}u(t) + \delta(t) \quad (57)$$

$$y_d(t) = \tilde{C}\tilde{x}(t), \quad (58)$$

where $\tilde{A} = \begin{bmatrix} A & 0 \\ 4\Gamma_0^{-1}C & -2\Gamma_0^{-1} \end{bmatrix}$, $\tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$, $\tilde{C} = [-C \quad I]$, and $\delta(t) = \begin{bmatrix} 0 & 0 \\ 4\Delta C & -2\Delta \end{bmatrix} \tilde{x}(t)$.

In the $M - \Delta$ loop representation of the system (57)-(58), the system matrices are given by

$$A_{cl} = \begin{bmatrix} \tilde{A} - \tilde{B}K & \tilde{B}K \\ 0 & \tilde{A} - L\tilde{C} \end{bmatrix}, \quad B_{cl} = [0 \quad 4I \quad 0 \quad 4I]^T, \\ C_{cl} = [I \quad -\frac{1}{2}I \quad 0 \quad 0].$$

By exploring the largest eigenvalues of B_{cl} and C_{cl} , we have tight bounds on $B_{cl}B_{cl}^T \leq 32I$ and $C_{cl}^TC_{cl} \leq \frac{5}{4}I$. Therefore, the parameters of the input and output selection metrics are given by $b = 32$ and $c = 5/4$. Hence, the input selection metric for system with uncertain time delays is

$$f(S) = \sum_{i: \operatorname{Re}(\lambda_i) \geq -2\sigma\sqrt{10}} \operatorname{dist}^2(v_i, \operatorname{span}(W(S))), \quad (59)$$

and the output selection metric is

$$g(T) = \sum_{i: \operatorname{Re}(\lambda_i) \geq -2\sigma\sqrt{10}} \operatorname{dist}^2(v'_i, \operatorname{span}(M(T))). \quad (60)$$

For the uncertain time delay, Algorithm 1 uses a much higher eigenvalue threshold ($-2\sigma\sqrt{10}$) compared to the additive uncertainty in A ($-\sigma\sqrt{2}$).

D. Structured Uncertainty in Matrix A

In previous analysis, it has been assumed that the each entry of the system matrix is perturbed independently. In some problems (e.g., constant output feedback with uncertainty in the gain matrix), there are only a small number of uncertain parameters, but each uncertain parameter affects multiple entries of the system matrix [38]. In what follows, we assume that the uncertainty matrix Δ has the following structure:

$$\Delta = \sum_{i=1}^N \delta_i E_i,$$

where E_i are constant matrices and δ_i are uncertain parameters varying in the interval $[-\epsilon_i, \epsilon_i]$ for each i .

When such a structured uncertainty Δ is added to the matrix A , the system has the dynamics

$$\dot{x}(t) = (A + \Delta)x(t) + Bu(t) \\ y(t) = Cx(t).$$

By involving the state estimation \hat{x} and feedback control $u = -K\hat{x}$, the closed-loop dynamics (7) of the above system have the form

$$\dot{\bar{x}} = A_{cl}\bar{x} + \left(\sum_{i=1}^N \delta_i \bar{E}_i \right) \bar{x}(t), \quad \text{where } \bar{E}_i = \begin{bmatrix} E_i & 0 \\ E_i & 0 \end{bmatrix}. \quad (61)$$

A sufficient condition guaranteeing the stability of the system (61) is given by the following lemma.

Lemma 9 ([38]). Define $P_i = (\bar{E}_i^T P + P\bar{E}_i)/2$, where P is the unique solution of the Lyapunov equation

$$PA_{cl} + A_{cl}^T P + 2I = 0. \quad (62)$$

The closed-loop system (61) is asymptotically stable if the following condition holds:

$$|k_j| < 1 / \left\| \sum_{i=1}^N P_i \right\|_{\infty} \text{ for all } j. \quad (63)$$

The following lemma presents a sufficient condition on the solution P of the Lyapunov equation (62) to satisfy the stability condition (63).

Lemma 10. Let σ be the largest singular value of $(\sum_{i=1}^N \bar{E}_i)$. Assume $k_i \in [-\epsilon_i, \epsilon_i]$ for all i . Then the Lyapunov equation (62) has solution P satisfying the condition (63) if

$$P \prec \frac{1}{\sigma\epsilon} I \quad (64)$$

where $\epsilon = \max_i \{\epsilon_i\}$.

Proof. Given $k_i \in [-\epsilon_i, \epsilon_i]$, we have $|k_i| \leq \epsilon_i$. Define $\epsilon = \max_i \{\epsilon_i\}$. Then the condition (63) is satisfied if

$$\left\| \sum_{i=1}^N P_i \right\|_{\infty} < \frac{1}{\epsilon}. \quad (65)$$

Since the matrix P constructing P_i is the solution of the Lyapunov equation (62), P is symmetric positive definite and hence

$$\begin{aligned} \left\| \sum_{i=1}^N P_i \right\|_{\infty} &= \left\| \sum_{i=1}^N (\bar{E}_i^T P + P\bar{E}_i)/2 \right\|_{\infty} \\ &\leq \frac{1}{2} \left\| \sum_{i=1}^N \bar{E}_i^T P \right\|_{\infty} + \frac{1}{2} \left\| \sum_{i=1}^N P\bar{E}_i \right\|_{\infty} \\ &\leq \left\| \sum_{i=1}^N \bar{E}_i \right\|_{\infty} \|P\|_{\infty}. \end{aligned}$$

When the largest singular value of $(\sum_i^N \bar{E}_i)$ is denoted by σ , the inequality (65) holds if

$$\|P\|_{\infty} < \frac{1}{\sigma\epsilon}.$$

For symmetric positive definite matrix P , it is equivalent to requiring

$$P \prec \frac{1}{\sigma\epsilon} I,$$

which completes the proof. \square

By Lemma 9 and 10, if $P \prec (1/\sigma\epsilon)I$, then the system (61) is stable. Assuming A_{cl} is Hurwitz, a positive definite solution P to the Lyapunov equation (62) is given by [18],

$$P = \int_0^{\infty} e^{A_{cl}^T t} (2I) e^{A_{cl} t} dt.$$

Let $\lambda(A_{cl})$ be the largest eigenvalue of A_{cl} . Then the sufficient condition to guarantee system stability in the presence of structured uncertainty Δ becomes

$$\int_0^\infty e^{\lambda(A_{cl})t} (2) e^{\lambda(A_{cl})t} dt < \frac{1}{\epsilon\sigma}.$$

Simplifying the above expression, we obtain

$$\lambda(A_{cl}) < -\epsilon\sigma \quad (66)$$

where $\epsilon = \max_i \{\epsilon_i\}$ and $\sigma = \|\sum_{i=1}^N \bar{E}_i\|_\infty$.

The condition (66) on eigenvalues of A_{cl} is similar to the condition (20). Following the rest steps of our approach, we have the metrics

$$f(S) = \sum_{i: \operatorname{Re}(\lambda_i) \geq -\epsilon\sigma} \operatorname{dist}^2(v_i, \operatorname{span}(W(S))) \quad (67)$$

for the input selection and

$$g(T) = \sum_{i: \operatorname{Re}(\lambda_i) \geq -\epsilon\sigma} \operatorname{dist}^2(v'_i, \operatorname{span}(M(T))) \quad (68)$$

for the output selection.

Since the analysis for structured uncertainty does not depend on the $M - \Delta$ transformation, the eigenvalue threshold ($-\epsilon\sigma$) that is used in the Algorithm 1 has a different structure compared to previous cases ($-\sigma\sqrt{bc}$).

VI. NUMERICAL STUDY

This section presents numerical results of our proposed input and output selection approach on the IEEE 39-bus power system. We adopt the Linear Quadratic Regulator (LQR) for controller design and study the system stability in the presence of uncertainties of different types and magnitudes. We illustrate the optimality of our method by comparing with the existing geometric indices based selection strategy [39].

A. System Setup

We consider the wide-area damping control in power systems to ensure small signal stability [35]. Given an unstable initial operating point, the goal of the controller is to damp inter-area oscillations between generators and stabilize rotor speeds at each generator. Each generator's Power System Stabilizers (PSS) is an input node and the control signals are to adjust excitation voltages of each generator, which eventually impact the power output. The control outputs consist of measurable parameters throughout the system, including power outputs of generators and power flows over transmission lines. The test system in this numerical study has 10 input nodes and 56 output nodes.

The initial power system, including system topology and generator configurations, is obtained from the IEEE 39-bus power system data [11]. We create initial operating points with different unstable modes by varying the overall load level and gain parameters of PSS at each generator.

Linearizing the system dynamics at any initial operating point gives a linear approximation of the control system with matrices A, B, C in the state-space form (1)-(2). In this

control system, uncertainties may arise due to load changing, unexpected transmission line tripping, and time delays.

The problem we studied is to select a minimal set of generators (inputs) and observations (outputs) to be involved in the wide-area damping control, which ensures the existence of a stabilizing controller that can damp all generator rotor oscillations. We test our proposed selection algorithm and present simulation results in two subsections. In the first, we demonstrate the minimal input and output selections for nominal systems and compare our proposed selection approach with a method based on geometric indices. In the second subsection, we show how different types and magnitudes of uncertainties affect the minimal selections. We also show that the proposed selection approach guarantees the system stability when uncertainties are within the assumed bound, i.e., $\|\Delta\|_\infty \leq \sigma$.

The geometric indices based input/output selection approach is implemented as follows. The algorithm first computes the geometric measures of controllability $m_{ci}(k)$ and observability $m_{oi}(k)$ for each unstable mode k and each input/output i as follows [39],

$$m_{ci}(k) = \frac{b_i^T \psi_k}{\|\psi_k\| \|b_i\|}, \quad m_{oi}(k) = \frac{c_i \phi_k}{\|\phi_k\| \|c_i\|},$$

where b_i is the i th column of matrix B and c_i is the j th row of matrix C ; ψ_k and ϕ_k are left and right eigenvectors of matrix A respectively corresponding to the eigenvalue λ_k . Then the algorithm iteratively selects inputs and outputs in a greedy manner. At each iteration, the input and output with the highest controllability and observability indices are selected and added to the sets S and T , respectively. The algorithm terminates when both of the stabilizability and detectability constraints are satisfied, i.e., $f(S) = 0$ and $g(T) = 0$.

B. Input and Output Selection for Nominal Systems

In this test case, we assume that there is no perturbation in the power system, $\sigma = 0$. We test our proposed selection algorithm for scenarios when there are different numbers of unstable modes. Unstable modes of a power system are typically caused by load shifting and improper functioning of power system stabilizers. We generate operating points with up to seven unstable modes by varying the load level randomly between 80% – 150% of the initial value, as well as scaling the gain parameters of all power system stabilizers to negative 116.2%.

Fig. 2(a) shows the system topology considered in this study, where one unstable mode occurs due to uncoordinated behaviors at generator 2 and 3. Our proposed algorithm selects generators $\{3, 5, 9\}$ as control input nodes to ensure system stabilizability.

A comparison between the proposed input/output selection and the geometric indices based selection is shown in Fig. 2(b) and 2(c). Each data point is the average number of selections at ten random operating points with the same number of unstable modes.

We observed that the average number of input and output nodes required by wide-area damping control increases as the

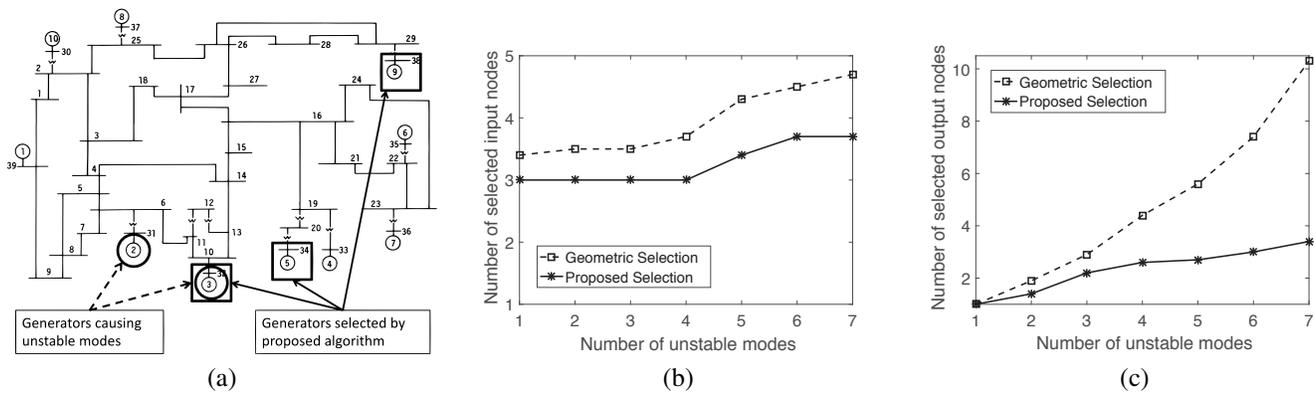


Fig. 2: Simulation study of our proposed approach to minimal input and output selection for power system stability. (a) IEEE 10-generator 39-bus power system topology considered in this study. Circled generators have initial rotor angles differing from the rest, causing unstable system modes. Generators selected by our proposed algorithm are indicated by squares. (b) Comparison of the average numbers of controlled generators (input nodes) required to meet the stabilizability condition ($f(S) = 0$) using the proposed approach and the method based on geometric indices. (c) Comparison of average numbers of observations (output nodes) for the proposed approach and the geometric indices based method to achieve the detectability condition ($g(T) = 0$).

number of unstable modes increases. We also found that the proposed selection algorithm requires fewer inputs and outputs for designing a stabilizing controller.

C. Input and Output Selection for Uncertain Systems

In this test case, we study how uncertainties impact our selection decisions at a fixed operating point, and evaluate the system robustness within our proposed selection framework. The initial operating point is obtained by using the default load level but scaling up the gains of all power system stabilizers by 16.2% while two gains are changed to negative.

Tables I and II demonstrate the input and output selections for different types of uncertainties at the same operating point. The metrics established in Section V are used in our proposed algorithm for different uncertainty cases. For structured uncertainty, we assume that the uncertain parameters k_i are bounded by $k_i \in [-1, 1]$ and hence $\epsilon = \max_i \{|k_i|\} = 1$. We compare selections for different uncertainties as the largest singular value σ increases from 0 to 0.35. We observed that the number of selections required to satisfy the stabilizability/detectability conditions increases as the uncertainty magnitude increases. It has also been observed that the number of selections for multiplicative uncertainty increases faster than other uncertainty cases, which indicates that the proposed selection approach is more sensitive to the multiplicative uncertainty. This is because that, comparing the thresholds $\text{Re}(\lambda) \leq -\sigma\sqrt{bc}$ for different uncertainties, the multiplicative uncertainty requires a much larger number of “desired modes” (proportional to σ) to lie in the controllability/observability subspaces.

In what follows, results on evaluating the system robustness to uncertainties are presented. For any set of input and output nodes, we consider the following LQR design of a feedback control $u = -Kx$, which is to minimize the cost function defined by

$$J(u) = \int_0^{\infty} (2x^T x + u^T u) dt.$$

TABLE I: Comparing inputs selected by the proposed algorithm under different uncertainties.

Uncertainty bound, σ	0	0.02	0.04	0.06	0.35
Outputs for additive uncertainty	{4, 5, 10}	{3, 5, 9}	{3, 5, 9}	{3, 5, 9}	{2, 3, 5, 9}
Outputs for multiplicative uncertainty	{4, 5, 10}	{2, 3, 5, 9}	{3, 7, 8, 10}	{3, 7, 8, 10}	{3, 7, 8, 10}
Outputs for uncertain delays	{4, 5, 10}	{3, 5, 9}	{3, 5, 9}	{2, 3, 5, 9}	{3, 7, 8, 10}
Outputs for structured uncertainty	{4, 5, 10}	{3, 5, 9}	{3, 5, 9}	{3, 5, 9}	{2, 3, 5, 9}

TABLE II: Comparing outputs selected by the proposed algorithm under different uncertainties.

Uncertainty bound, σ	0	0.02	0.04	0.06	0.35
Outputs for additive uncertainty	{26, 35}	{25, 38}	{22, 28, 45}	{22, 28, 45}	{2, 19, 43}
Outputs for multiplicative uncertainty	{26, 35}	{2, 19, 43}	{13, 21, 46}	{13, 21, 46}	{13, 21, 46}
Outputs for uncertain delays	{26, 35}	{19, 20, 54}	{19, 20, 54}	{2, 19, 43}	{13, 21, 46}
Outputs for structured uncertainty	{26, 35}	{25, 38}	{25, 38}	{22, 28, 45}	{2, 19, 43}

To evaluate if a set of selections can guarantee system stability, we randomly generate 1000 uncertain matrices Δ , with the same largest singular value σ , and add to matrix A . For each case, we apply the same LQR controller designed from the nominal system parameters. We measure the robustness of this control system by the percentage of cases that is stable out of these 1000 random trials.

Selections by Algorithm 1 with threshold $\sigma = 0$ are used as a baseline in comparison to the proposed selection approach that guarantees stability. When $\sigma = 0$, the selected inputs ensure stability of the nominal system but do not guarantee robustness to any uncertainties, while $\sigma \neq 0$ results in a selection for stabilizing control assuming $\|\Delta\|_{\infty} \leq \sigma$. At this operating point, the Algorithm 1 with threshold $\sigma = 0$ selects generators {4, 5, 10} as inputs and observations {26, 35} as outputs. For additive uncertainties in matrix A assuming largest singular values less than $\sigma = 40$, our proposed algorithm selects generators {3, 7, 8, 10} as input nodes and

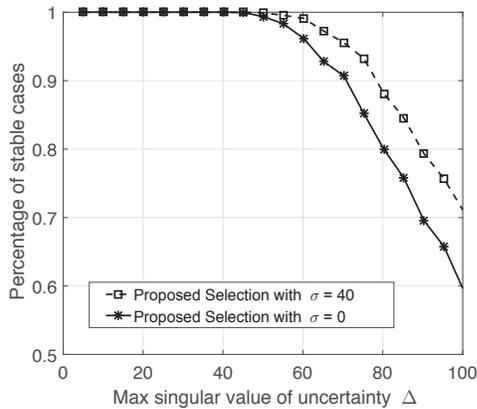


Fig. 3: Comparison of percentages of cases that the controlled system is stable in the presence of additive uncertainty Δ using the proposed selection with $\sigma = 0$ and the selection with $\sigma = 40$. Each data point has 1000 random trials in Δ . The proposed algorithm with robustness consideration ($\sigma = 40$) is stable in a higher percentage of cases compared to the selection without considering uncertainties ($\sigma = 0$).

observations $\{13, 21, 46\}$ as outputs.

Fig. 3 shows a comparison of system robustness between the two systems with input/output nodes selected by $\sigma = 0$ and $\sigma = 40$. Each data point is the percentage of stable cases in 1000 trials of random additive uncertainty Δ with the same largest singular value. We observed that when uncertainties have magnitudes within the assumption, $\|\Delta\|_\infty \leq 40$, then the controller based on the selections for $\sigma = 40$ ensures system stability in all cases. When uncertainties have largest singular values over the bound, for example $\|\Delta\|_\infty = 53 > 40$, both selections fail to guarantee stability. Our approach with robust consideration $\sigma = 40$, however, achieves a system with more stable cases (93.25%) compared to the selection based on $\sigma = 0$ (85.20%).

VII. CONCLUSIONS

In this paper, we studied the problem of selecting minimal inputs and outputs to ensure existence of a stabilizing controller for an uncertain linear system. We derived sufficient conditions for input selection and output selection that guarantee system stabilizability and detectability respectively in the presence of uncertainties, requiring that a subset of system modes lie in the controllability/observability subspace. We formulated the problem of selecting minimal inputs/outputs to ensure that the distance from these desired modes to the span of the controllability/observability Gramian is zero. We showed that both input and output selection problems have bounded submodularity ratios. We developed computationally efficient algorithms with provable optimality bounds for minimal input selection, minimal output selection, and joint input and output selections, all ensuring the system stability in the presence of uncertainties. We also analyzed four types of uncertainties under our framework, including additive system uncertainty, multiplicative uncertainty, output time delay, and structured uncertainty. Our approach was validated through

numerical study on the IEEE 39-bus power system with comparison to existing input/output selection methods.

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APPENDIX

Proof of the statement "If $0 \prec X \prec \epsilon I$, then $X^2 \prec \epsilon^2 I$ ":

For any $X \succ 0$, we have $\epsilon^2 I - X^2 = (\epsilon I - X)(\epsilon I + X)$, where $\epsilon I - X$ is positive definite by assumption that $X \prec \epsilon I$ and $\epsilon I + X \succ X \succ 0$ is also positive definite.

Both $A = \epsilon I - X$ and $B = \epsilon I + X$ are symmetric positive definite and commute. Hence we have $AB = \epsilon^2 I - X^2$ also positive definite, which implies $X^2 \prec \epsilon^2 I$. \square

Proof of Lemma 5: Let $\{q_1, \dots, q_{n_1}\}$ be a maximal set of linearly independent columns of the controllability matrix \mathcal{C} , and define a matrix P such that $P^{-1} = (q_1 \cdots q_{n_1} \cdots q_n)$, where q_{n_1+1}, \dots, q_n can be arbitrarily chosen so long as P^{-1} is nonsingular. Under the transformation $z = Px$, the system can be written as $\dot{z} = \bar{A}z + \bar{B}u$ or more specifically

$$\begin{pmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ 0 \end{pmatrix} u(t),$$

where the n_1 -dimensional subsystem $\dot{z}_1 = A_{11}z_1 + B_1u$ is controllable (Chen [18, Theorem 6.6]).

Suppose first that λ is an eigenvalue where the corresponding eigenvector v lies in the span of the controllability matrix, i.e., $v \in \text{span}(\mathcal{C}) = \text{span}([B_s \ AB_s \ A^2 B_s \ \dots \ A^{n-1} B_s])$. We

show that λ is an eigenvalue of A_{11} , and hence all eigenvalues with $\text{Re}(\lambda) \geq \lambda_1$ are eigenvalues of A_{11} by assumption.

Indeed, letting $w = Pv$, we have that

$$\bar{A}w = PAP^{-1}w = PAP^{-1}Pv = PAv = \lambda Pv = \lambda w,$$

establishing that w is an eigenvector of \bar{A} corresponding to λ . Furthermore, we have that $v = P^{-1}w$, and hence w is the representation of v with respect to the columns of P^{-1} . Since v is in the span of \mathcal{C} , it also lies in the span of $\{q_1, \dots, q_{n_1}\}$, and thus $w_i = 0$ for $i > n_1$. We then have $w = (w_1 \ 0)^T$, where w_1 is an eigenvector of A_{11} corresponding to eigenvalue λ .

A similar argument to the above establishes that all eigenvalues corresponding to "undesired modes" ($\text{Re}(\lambda) \geq \lambda_1$) are eigenvalues of A_{11} , and that the system (A_{11}, B_1) is controllable. Hence, we can design a feedback controller K such that the system $A_{11} - B_1K$ has eigenvalues in the desired region ($\text{Re}(\lambda) < \lambda_1$) [18, Theorem 8.3], while the remaining eigenvalues are unchanged ($\text{Re}(\lambda) < \lambda_1$). Applying the transformation P and using the fact that the eigenvalues of the system are invariant under a similarity transform yields the desired result. \square



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