2016 IEEE 55th Conference on Decision and Control (CDC)
ARIA Resort & Casino
December 12-14, 2016, Las Vegas, USA

MinGen: Minimal Generator Set Selection for Small Signal Stability in Power Systems: A Submodular Framework

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***Abstract*— Increased power transfers over a wide geographical area impact the small-signal stability of the power system, defined as the ability to damp oscillations between generators in different geographical areas. Wide-area control architectures have been proposed to control the generators in order to mitigate these oscillations. Ensuring that all unstable oscillating modes are removed depends on selecting a subset of generators to participate in wide-area control, which is inherently a discrete optimization problem that in the current literature does not have any solution algorithms with provable stability guarantees. In this paper, we present MinGen, a submodular optimization framework for generator selection for small-signal stability. We prove that small-signal stability is achieved when the unstable modes are controllable and observable from the set of selected generators, and map these properties to submodular constraints. We develop a computationally efficient and submodular MinGen algorithm with provable optimality bounds for generator selection, which can be generalized to enhance robustness to communication link failures. We evaluate our approach via a numerical study on the IEEE New England test case.**

I. Introduction

Growing energy demand and integration of renewable energy sources have led to a steady increase in power transfers across wide geographical areas [1]. Such power transfers can create low-frequency phase oscillations between generators in different regions, which, when poorly damped, cause the phase angles between generators to increase beyond the stability limit [2]. The ability of the power system to damp these oscillations before angle separation and outages occur is defined as the *small-signal stability* of the power system. Small-signal instability has resulted in generator tripping and catastrophic power outages, such as the 1996 Western United States blackout [3]. Ensuring small-signal stability has been identified as one of the critical power system stability challenges by the North American Synchrophasor Initiative (NASPI) [4].

The current approach to ensuring small-signal stability of the power system is through local control at generators using power system stabilizers (PSS) [2]. More recently, as increases in power transfers have made the stability of inter-area modes increasingly unpredictable, hierarchical, wide-area control mechanisms have been proposed in the power system literature [1], [5], [6]. Under the hierarchical approach, local PSSs are supplemented by a wide-area controller that receives state information from buses in multiple areas, evaluates the stability of the overall system, and sends control inputs to a subset of generators in order to ensure stability. This control methodology is enabled by the availability of real-time state information from Phasor Measurement Units (PMUs) and the integration of reliable, low-latency communications [7].

State of the art design techniques for wide-area control are based on linearizing the power system dynamics and identifying inter-area oscillations as unstable modes of the linear dynamics [1], [5], [6]. The goal of the wide-area control is to drive the system state to the equilibrium. A key step is selecting a subset of generators to participate in the wide-area control by receiving control signals from the controller and sending state measurements that are used to compute the control action [1], [5]. Selecting such a subset, however, is inherently a discrete (combinatorial) optimization problem, and hence is computationally intractable for large-scale power systems unless additional problem structure can be identified.

Current approaches for selecting generators instead use heuristic techniques that do not guarantee stability, and require the additional post-processing step of synthesizing a controller using the selected generators, evaluating the stability properties, and re-selecting the generators if stability is not satisfied. At present, a systematic approach to selecting a minimum-size set of generators with provable guarantees on small-signal stability is an open problem.

In this paper, we present MinGen, an analytical framework for selecting a set of generators for wide-area damping control. Our key insight is that the discrete optimization problem of selecting a set of generators has an inherent submodular structure, which can be exploited to develop algorithms with provable stability guarantees and bounds on the number of generators required for wide-area control.

We develop our framework in two components. The first component ensures that there exists a control input signal to the selected generators to stabilize the system, which we show is equivalent to controllability of the unstable modes. The second component ensures that the state measurements sent from the generators to the controller are sufficient to compute the control signal, which we show is equivalent to observability of the unstable modes. We make the following specific contributions:

- We show that a sufficient condition for a set of generators to ensure small-signal stability is that the distance

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This work was supported by NSF grant CNS-1544173.

between the unstable modes and the span of the controllability Gramian should be zero. We prove that the distance to the span of the controllability Gramian is supermodular as a function of the set of generators that are selected.
- We prove that a set of generator state measurements enables computation of a stabilizing control signal if the null space of the observability Gramian is contained in the span of the stable modes. We then map this condition to a submodular constraint by proving that the rank of the null space is supermodular as a function of the set of generators.
- We formulate the problem of selecting a minimum-size set of generators for wide-area control by combining the submodular constraints for controllability and observability of the unstable modes, and show that the submodular structure of the two problems is preserved in the joint problem. We develop efficient algorithms for selecting a set of generators and derive provable optimality bounds by exploiting the supermodularity of the joint objective function.
- We generalize our proposed approach to develop algorithms that guarantee stabilizability in the presence of communication link failures.
- We evaluate our approach through a numerical study on the IEEE New England Test System [8]. Our numerical results show that the proposed approach requires fewer generators to guarantee controllability and observability compared to existing techniques [9].

The paper is organized as follows. Section II reviews the related work. Section III presents the power system model. Section IV contains our submodular optimization approach to selecting generators and measurements for wide-area control. Section V presents numerical results. Section VI concludes the paper.

II. Related Work

Small-signal stability is a critical property of power systems, and hence has received extensive research attention over the past several decades [2], [5], [6], [7], [9]. Recently, wide-area control methodologies that use real-time state information from PMUs have been proposed to mitigate inter-area oscillations [1], [5]. These related works have identified the selection of generators to implement wide-area control as a key component of the design, and have proposed selecting generators based on geometric controllability and observability indices [1]. A related approach is sparsity-promoting control, which minimizes the level of communication overhead required to implement a controller by adding a sparsity-inducing l_1-norm to the controller objective function [10]. Techniques for l_0-optimization were proposed for quadratic model predictive control in [11]. These current approaches, however, do not guarantee that the number of generators and measurements selected is within a provable bound of the minimum size.

The control community has investigated the related problem of selecting a minimum-size set of control inputs to ensure controllability of an LTI system system [12], [13]. Controllability is a sufficient but not necessary condition for a set of generators to ensure small-signal stability. Indeed, while controllability implies that all modes of the system are controllable, small-signal stability only requires the unstable system modes to be controllable [6]. Hence, selecting a set of generators to satisfy controllability of all modes, when controllability of only the unstable modes is sufficient, could lead to a larger set of generators with additional communication overhead, time delays, and controller complexity. To the best of our knowledge, the problem of selecting input nodes to ensure that only the unstable modes are controllable with provable guarantees, which is the case for small-signal stability in power systems, has not been studied. In [14], an efficient approximation algorithm was proposed for the minimal reachability problem, based on the related metric of the span of the controllability matrix. The submodular structure of the problem, however, was not explored.

III. Power System Model

We consider a multi-machine power system with n generators, where each generator's dynamics are represented by its swing equation and locally closed-loop controls including excitation system, Automatic Voltage Regulator (AVR) and PSS. Generators are coupled through power flows. Using a Kron reduction [2], all PQ buses that are not directly connected to a generator can be eliminated and the system topology can be reduced to a fully connected circuit graph where nodes represent generators. The resistance of the transmission lines is neglected. The Kron-reduced admittance matrix Y has pure imaginary entries $-\mathbf{j}Y_{ij}$, where $\mathbf{j} = \sqrt{-1}$, that describe the interactions between generators i and j. Therefore, the power flow from generator i to j is given as

$$\begin{aligned} P_{ij} &= \Re\left[E_i e^{\theta_i}\left(-\mathbf{j}Y_{ij}(E_i e^{\mathbf{j}\theta_i} - E_j e^{\mathbf{j}\theta_j})\right)^*\right] \\ &= Y_{ij}E_iE_j\sin(\theta_i - \theta_j) \end{aligned} \tag{1}$$

where E_i is the magnitude of the internal electromotive force (e.m.f) of generator i, which is assumed to remain constant at the pre-disturbance value; θ_i is the rotor angle of generator i. The generator internal e.m.f magnitudes E_i, E_j are controlled by the exciter, AVR, and PSS of each generator.

At each generator i, linearizing the swing dynamics around an operating point, denoted by $\{\theta_i^0, i = 1, \ldots, n\}$, gives

$$\Delta\dot{\theta}_i = \Delta\omega_i \tag{2}$$

$$\begin{aligned} \Delta\dot{\omega}_i = \frac{1}{M_i}\Bigg(&- D_i\Delta\omega_i \\ &- \sum_{j=1; j\neq i}^{n} Y_{ij}E_iE_j\cos(\theta_i^0 - \theta_j^0)(\Delta\theta_i - \Delta\theta_j)\Bigg) \end{aligned} \tag{3}$$

where ω_i represents the rotor frequency of generator i; M_i and D_i are the inertia and damping coefficients. The mechanical power injections P_m are assumed to be constants.

Considering n generators at the same time, the swing dynamics are given by

$$\begin{bmatrix}\Delta\dot{\theta}\\ \Delta\dot{\omega}\end{bmatrix} = \begin{bmatrix}0_{n\times n} & I_{n\times n}\\ M^{-1}L & -M^{-1}D\end{bmatrix}\begin{bmatrix}\Delta\theta\\ \Delta\omega\end{bmatrix} \tag{4}$$

where $\Delta\theta = [\Delta\theta_1, \Delta\theta_2, \ldots, \Delta\theta_n]^T$ and $\Delta\omega = [\Delta\omega_1, \Delta\omega_2, \ldots, \Delta\omega_n]^T$; M and D are diagonal matrices with diagonal elements M_i and D_i respectively; L is a dense matrix with off-diagonal elements $L_{ij} = Y_{ij}E_iE_j\cos(\theta_i^0 - \theta_j^0)$ and diagonal elements $L_{ii} = -\sum_{j=1;j\neq i}^{n} L_{ij}$.

Considering other local dynamics such as AVR and PSS, we specify the internal e.m.f magnitude changes ΔE_i at each generator as a function of local state variables $\Delta E_{fd_i}, \Delta E_{f_i}$ and the central control signal u_i. As a result, the change in rotor frequency $\Delta\omega_i$ is also affected by these local variables in the following way: A new term $-\frac{K_2}{M_i}\Delta E_{f_i}$ is added to $\Delta\omega_i$'s dynamics (3), while ΔE_{fd_i} and ΔE_{f_i} have dynamics

$$\begin{aligned}\Delta\dot{E}_{fd_i} &= -\frac{1}{\tau_{Ai}}\left[K_{Ai}K_{5i}\Delta\theta_i + \Delta E_{fd_i} + K_{Ai}K_{6i}\Delta E_{f_i}\right]\\ &\quad + \frac{K_{Ai}}{\tau_{Ai}}(u_{pss_i} + u_i)\\ \Delta\dot{E}_{f_i} &= \frac{1}{\tau_{doi}}\left[-K_{4i}\Delta\theta_i + \Delta E_{fd_i} - \frac{1}{K_{3i}}\Delta E_{f_i}\right]\end{aligned}$$

where $\tau_{Ai}, \tau_{doi}, K_{Ai}, K_{2i}, K_{3i}, K_{4i}, K_{5i}, K_{6i}$ are local AVR parameters; $u_{pss_i} = K_{pss_i}\Delta\omega_i$ is the local PSS closed-loop control input with feedback parameter K_{pss_i}; u_i is the central control signal sent to local PSS.

Define local state vectors $\Delta E_{fd} = [\Delta E_{fd_1}, \ldots, \Delta E_{fd_n}]^T$ and $\Delta E_f = [\Delta E_{f_1}, \ldots, \Delta E_{f_n}]^T$. The system model in (4) can then be extended to include the local AVR and PSS dynamics. Hence we have

$$\begin{bmatrix}\Delta\dot{E}_{fd}\\ \Delta\dot{E}_f\\ \Delta\dot{\theta}\\ \Delta\dot{\omega}\end{bmatrix} = A\begin{bmatrix}\Delta E_{fd}\\ \Delta E_f\\ \Delta\theta\\ \Delta\omega\end{bmatrix} + \begin{bmatrix}T_A^{-1}K_A\\ 0_{n\times n}\\ 0_{n\times n}\\ 0_{n\times n}\end{bmatrix}\begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix} \tag{5}$$

where

$$A = \begin{bmatrix}-T_A^{-1} & T_A^{-1}K_AK_6 & T_A^{-1}K_AK_5 & T_A^{-1}K_AK_{pss}\\ T_{do}^{-1} & -T_{do}^{-1}K_3^{-1} & -T_{do}^{-1}K_4 & 0_{n\times n}\\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & I_{n\times n}\\ 0_{n\times n} & -M^{-1}K_2 & M^{-1}L & -M^{-1}D\end{bmatrix}$$

and T_A, T_{do}, K_A, K_2, K_3, K_4, K_5, K_6, K_{pss} are diagonal matrices with diagonal elements τ_{Ai}, τ_{doi}, K_{Ai}, K_{2i}, K_{3i}, K_{4i}, K_{5i}, K_{6i}, K_{pss_i} respectively.

The full system model (5) can be generalized into the following state-space form:

$$\dot{x} = Ax + Bu \tag{6}$$
$$y = Cx \tag{7}$$

where $x = [\Delta E_{fd}^T, \Delta E_f^T, \Delta\theta^T, \Delta\omega^T]^T \in R^{4n\times 1}$ is the state vector; $y \in R^{m\times 1}$ is a vector of those states in x that are observable; $u = [u_1, \ldots, u_n]^T \in R^{n\times 1}$ is the input vector whose entries u_i are control signals sent by the central controller to generator i; $A \in R^{4n\times 4n}$ has a set Γ of modes with corresponding orthonormal eigenvectors $v_i, i \in \Gamma$; $B \in R^{4n\times n}$ and $C \in R^{m\times 4n}$ are rectangular diagonal matrices. We denote $B = [b_1, \ldots, b_n]$ where b_i is the ith column of B, with nonzero entry b_{ii}.

The power system is stable when all eigenvalues of matrix A have only non-positive real parts. Small signal instability arises when A has eigenvalues with positive real parts under some operating conditions. This occurs when generators in different areas oscillate against each other. The central control signal u attempts to stabilize the system by driving current state x_0 to the origin.

Finally, recall that a set function $f : 2^V \rightarrow R$ is *submodular* if, for any $S \subseteq T \subseteq V$ and any $v \in V \setminus T$,

$$f(S \cup \{v\}) - f(S) \geq f(T \cup \{v\}) - f(T).$$

A function f is supermodular if $-f$ is submodular.

IV. MinGen Generator Selection Framework

This section presents the MinGen framework for selecting a minimum-size set of generators for wide-area control. The generator selection problem has two components: ensuring that the generators are sufficient to drive the system state to equilibrium, and ensuring that the control signal can be computed using state information sent from the selected generators. We map each component to a submodular constraint on the set of generators. Based on the submodularity we establish for each constraint, we formulate the problem of selecting a set of generators to satisfy both the control and observability conditions.

A. Constraints on Generator Control Signals

When a subset of generators $S \subseteq \Omega$ receives control inputs, the system dynamics (6) become

$$\dot{x}(t) = Ax(t) + B_su_s(t) \tag{8}$$

where the columns of B_s are given by $(b_i : i \in S)$. The selected set of generators S therefore impacts the control of the system by determining the columns of the B matrix. The goal of our proposed approach is defined as follows.

Definition 1: The system dynamics (8) satisfies reachability to 0 if, for any $\epsilon > 0$ and initial state x_0, there is a time t_1 and a control signal $\{u_s(t; x_0) : t \in [t_0, t_1]\}$ such that $||x(t_1)||_2 < \epsilon$.

The following lemma gives an equivalent condition for existence of a control input that drives the system to a desired state at some time t_1.

Lemma 1 ([15]): There exists a control u_s from state x_0 at time t_0 to state x_1 at time $t_1 > t_0$ if and only if $x_1 - e^{A(t_1-t_0)}x_0$ is in the column space of the controllability Gramian

$$W(S) = \int_{t_0}^{t_1} e^{A(t-t_0)}B_sB_s^Te^{A^T(t-t_0)}dt. \tag{9}$$

If η is a solution to $x_1 - e^{A(t_1-t_0)}x_0 = W(S)\eta$, then a control given by $u(t) = B_s^Te^{A^T(t_1-t)}\eta$ would make the desired transfer.

We now present a sufficient condition for Definition 1 based on Lemma 1.

Lemma 2: Definition 1 holds if, for any vector v satisfying $Av = \lambda v$ for some $\lambda \geq 0$, $v \in \text{span}(W(S))$.

Proof: Let $\epsilon > 0$ and suppose x_0 denotes the initial state. Define $\{v_1, \ldots, v_{4n}\}$ to be a basis of eigenvectors of A, let $\{\lambda_1, \ldots, \lambda_{4n}\}$ denote the corresponding eigenvalues, and suppose without loss of generality that $\lambda_i \geq 0$ for $i = 1, \ldots, l$ and $\lambda_i < 0$ for $i = (l+1), \ldots, 4n$.

Suppose $v_i \in \text{span}(W(S))$ for $i = 1, \ldots, l$. We can write $x_0 = \alpha_1 v_1 + \cdots + \alpha_{4n} v_{4n}$. Choose $t_1 > t_0$ such that $|e^{\lambda_i(t_1 - t_0)}\alpha_i| < \frac{\epsilon}{4n}$ for all $i = (l+1), \ldots, 4n$. Now, by Lemma 1, there exist control signals $\{u_i(t) : i = 1, \ldots, l, t \in [t_0, t_1]\}$ such that $x(t_1) = 0$ when the initial state is given by v_i. Define

$$u(t) = \sum_{i=1}^{l} \alpha_i e^{\lambda_i(t_1 - t_0)} u_i(t).$$

We then have

$$\begin{aligned} x(t_1) &= e^{A(t_1 - t_0)} x_0 + \int_{t_0}^{t_1} e^{A(t_1 - t)} B_s u(t)\ dt \\ &= \sum_{i=l+1}^{4n} \alpha_i e^{\lambda_i(t_1 - t_0)} v_i + \sum_{i=1}^{l} \Big[\alpha_i e^{\lambda_i(t_1 - t_0)} v_i \\ &\quad + \alpha_i e^{\lambda_i(t_1 - t_0)} \int_{t_0}^{t_1} e^{A(t_1 - t)} B_s u_i(t)\ dt\Big] \\ &= \sum_{i=l+1}^{4n} \alpha_i e^{\lambda_i(t_1 - t_0)} v_i. \end{aligned}$$

Hence, $||x(t_1)||_2$ is bounded by

$$||x(t_1)||_2 \leq \sum_{i=l+1}^{4n} ||\alpha_i e^{\lambda_i(t_1 - t_0)} v_i||_2 < \epsilon.$$

■

Motivated by Lemma 2, we define the metric $f(S)$ by

$$f(S) \triangleq \sum_{i \in \Gamma'} d^2(v_i, \text{span}(W(S))), \tag{10}$$

where $\Gamma' \subseteq \Gamma$ refers to the index set of unstable modes and $d(\cdot, \cdot)$ in (10) denotes distance in the Euclidean metric. The metric is equivalent to the distance between the eigenvectors corresponding to unstable modes and the span of the controllability Gramian, and, intuitively, is a measure of how close the unstable modes are to being controllable.

By Lemma 2, the condition $f(S) = 0$ is sufficient to ensure Definition 1. We next investigate the structure of $f(S)$.

B. Submodularity of Control Signal Constraint

We show that the function $f(S)$ has submodular structure. We first define a set operator below that will be used throughout the rest of this section.

Definition 2: If a set of vectors $\{w'_k, k \in T'\}$ are linearly independent and $S' \subseteq T'$, then we define the set operator $\ominus$ as

$$span\{w'_{k \in T'}\} \ominus span\{w'_{k \in S'}\} = span\{w'_{k \in T' \setminus S'}\}, \tag{11}$$

while $\bigoplus$ is defined as

$$span\{w'_{k \in T'}\} \bigoplus span\{w'_{k \in S'}\} = span\{w'_{k \in T' \cup S'}\}. \tag{12}$$

Theorem 1: The set function $f(S)$ as defined in (10) is supermodular.

The submodularity of a similar metric to $f(S)$ was identified in [16, Lemma 8.1], outside the scope of dynamical systems.

Before giving the proof of Theorem 1, we present two preliminary results that establish the main argument in proving the theorem.

Lemma 3: For any set of vectors $\{w_k, k \in S\}$,

$$span\{\sum_{k \in S} w_k w_k^T\} = span\left\{w_k : k \in S\right\}. \tag{13}$$

Proof: The proof can be found in the appendix. ■

Lemma 4: For any sets S and T with $S \subseteq T$ and any $v \notin T$,

$$\begin{aligned} &span\{W(T \cup \{v\})\} \ominus span\{W(T)\} \\ &\qquad \subseteq span\{W(S \cup \{v\})\} \ominus span\{W(S)\}, \end{aligned} \tag{14}$$

where the set subtraction $\ominus$ is defined in Definition 2.

Proof: By the definition of $W(S)$ in (9), the controllability Gramian can be approximated by a sum of matrices products as follows.

$$\begin{aligned} W(S) &= \int_{t_0}^{t_1} X(t) B_s B_s^T X(t)^T\ dt &(15) \\ &\approx \sum_{t_i = t_0}^{t_1} \sum_{k \in S} b_{kk}^2 w_k(t_i)(w_k(t_i))^T \Delta t, &(16) \end{aligned}$$

where $X(t) = e^{A(t - t_0)} \in R^{4n \times 4n}$ and $w_i(t)$ denotes the ith column of $X(t)$.

We then have

$$\begin{aligned} &span\{W(S \cup \{v\})\} \ominus span\{W(S)\} \\ &\quad = span\{W(S) + L\} \ominus span\{W(S)\}, &(17) \\ &span\{W(T \cup \{v\})\} \ominus span\{W(T)\} \\ &\quad = span\{W(T) + L\} \ominus span\{W(T)\}, &(18) \end{aligned}$$

where $L = \sum_{t_i = t_0}^{t_1} b_{vv}^2 w_v(t_i)(w_v(t_i))^T \Delta t$.

Since $W(S), W(T)$ and L are all positive linear combinations of $\{w_k(t_i) w_k(t_i)^T\}$, to prove (14), it suffices to show for a general set of vectors $\{w_k, k \in \Omega\}$, $S \subseteq T \subseteq \Omega$ and any vector $w_v \notin T$,

$$\begin{aligned} &span\{\sum_{k \in T} w_k w_k^T + w_v w_v^T\} \ominus span\{\sum_{k \in T} w_k w_k^T\} \\ &\subseteq span\{\sum_{k \in S} w_k w_k^T + w_v w_v^T\} \ominus span\{\sum_{k \in S} w_k w_k^T\}. \end{aligned} \tag{19}$$

By Lemma 3, $span\{\sum_{k \in S} w_k w_k^T\} = span\left\{w_{k \in S}\right\}$ for any set S. Thus, (19) can be simplified to

$$\begin{aligned} &span\left\{w_{k \in T}, w_v\right\} \ominus span\left\{w_{k \in T}\right\} \\ &\qquad \subseteq span\left\{w_{k \in S}, w_v\right\} \ominus span\left\{w_{k \in S}\right\}, \end{aligned} \tag{20}$$

which always holds, as a new vector w_v is more likely to increase the dimension of a smaller subspace $span\{w_{k\in S}\}$ compared to $span\{w_{k\in T}\}$. ■

Proof of Theorem 1: By using the property of $W(S)$ in Lemma 4, we next prove the distance function $f(S)=\sum_{i\in\Gamma'} f_i(S)$ is supermodular by showing $f_i(S)=d^2(v_i,\text{span}(W(S)))$ is supermodular for all i.

The whole space $R^{4n\times 4n}$ can be decomposed to

$$\begin{aligned} R^{4n} &= \left[R^{4n} \ominus span\left\{W(T\cup\{v\})\right\}\right] \\ &\bigoplus \left[span\left\{W(T\cup\{v\})\right\} \ominus span\left\{W(T)\right\}\right] \\ &\bigoplus span\left\{W(T)\right\} \\ &= span\{\bar{w}_{k\in\Omega\setminus(T\cup\{v\})}\} \bigoplus span\{\bar{w}_v\} \bigoplus span\{\bar{w}_{k\in T}\} \\ &= U_1 \bigoplus U_2 \bigoplus U_3 \end{aligned}$$

where $\{\bar{w}_{k\in S}\}$ is the orthonormal basis of $span\left\{W(S)\right\}$. Therefore, U_1,U_2,U_3 are an orthogonal decomposition of R^{4n}.

For any point $v_i\in R^{4n}$, we have

$$\begin{aligned} &f_i(T) = d^2\left(v_i,U_3\right) \\ &= d^2\left(v_i,R^{4n}\ominus U1\right) + d^2\left(x_0,R^{4n}\ominus U2\right) \\ &= d^2\left(v_i,span\left\{W(T\cup\{v\})\right\}\right) \\ &\;+ d^2\left(v_i,R^{4n}\ominus(span\left\{W(T\cup\{v\})\right\}\ominus span\left\{W(T)\right\})\right) \\ &= f_i(T\cup\{v\}) + f_i(\Omega\setminus v_T). \end{aligned}$$

Therefore, we have

$$f_i(T\cup\{v\}) - f_i(T) = -f_i(\Omega\setminus v_T). \tag{21}$$

By (14) in Lemma 4, we have

$$\begin{aligned} &f_i(\Omega\setminus v_T) \\ &= d^2\left(v_i,R^n\ominus\left(span\left\{W(T\cup\{v\})\right\}\ominus span\left\{W(T)\right\}\right)\right) \\ &\le d^2\left(v_i,R^n\ominus\left(span\left\{W(S\cup\{v\})\right\}\ominus span\left\{W(S)\right\}\right)\right) \\ &= f_i(\Omega\setminus v_S). \end{aligned}$$

Hence,

$$f_i(T\cup\{v\}) - f_i(T) \ge f_i(S\cup\{v\}) - f_i(S), \tag{22}$$

which implies $f_i(S)$ is supermodular. Hence the approximation of $f(S)$ introduced by (16) is submodular as a function of S. Taking the limit as Δt goes to zero establishes submodularity of $f(S)$, using the fact that a pointwise limit of submodular functions is submodular. ■

C. Constraints on Generator State Measurements

The control signal that is used to damp inter-area oscillations is computed based on the system state [6], [7]. These measurements are provided by PMUs at the generators. In order to ensure scalability, the number of generators sending PMU data should be as small as possible. This section presents a sufficient condition for state measurements sent by a set of generators to ensure that a stabilizing control signal can be computed.

We have that the measurements $y(t)$ are given by $y(t)=Cx(t)$, where C is an $m\times 4n$ diagonal matrix with rows $(c_i^T : i=1,\ldots,m)$. The set of states that are measured by generator k are indexed in the set G_k. Hence, the matrix C corresponding to a set of generators S is given by $C(S)=(c_i^T : i\in\bigcup_{k\in S} G_k)$.

Definition 3: A set of measurements S is sufficient to ensure stability if, for any $\epsilon>0$ and initial state x_0, there exists a time t_1 and an input $u(t)=f(y(t):t\ge t_0)$ such that $||x(t_1)||_2<\epsilon$.

The following lemma gives a preliminary step towards a sufficient condition for Definition 3.

Lemma 5: Let x_0 and $\hat{x}_0$ be actual and estimated initial states, respectively. Let $\epsilon>0$. There exists a time t_1 and a function f such that $u(t)=f(\hat{x}_0,t)$ results in $||x(t)||_2<\epsilon$ if

$$(x_0-\hat{x}_0)\in\text{span}(\{v_i : i\in\Gamma\setminus\Gamma'\}), \tag{23}$$

where $\Gamma\setminus\Gamma'$ denotes the indices of the stable modes.

Proof: If condition (23) holds, we have

$$x_0-\hat{x}_0 = \sum_{i\in\Gamma\setminus\Gamma'} \beta_i v_i \tag{24}$$

for some constants $\beta_i\in R$ and $\Gamma\setminus\Gamma'$ refers to the index set of stable modes.

By Lemma 1, there exists a control input $u(t)$ computed based on estimation $\hat{x}_0$ as $u(t)=B_s^T e^{A^T(t_1-t)}\hat{\eta}$ where $\hat{\eta}$ is a solution to $-e^{A(t_1-t_0)}\hat{x}_0=W(S)\hat{\eta}$. By exerting this control, the state $x(t)$ at time t_1 will be

$$x(t_1)=e^{A(t_1-t_0)}x_0+\int_{t_0}^{t_1} e^{A(t_1-t)}B_s u(t)\;dt. \tag{25}$$

Substituting $u(t)$ in the above equation, we get

$$\begin{aligned} x(t_1) &= e^{A(t_1-t_0)}x_0 + W(S)\hat{\eta} = e^{A(t_1-t_0)}(x_0-\hat{x}_0) \\ &= \sum_{i\in\Gamma\setminus\Gamma'} \beta_i e^{\lambda_i t_1} v_i. \end{aligned} \tag{26}$$

Since all eigenvalues of A corresponding to stable modes are non-positive, i.e. $\lambda_i\le 0, i\in\Gamma\setminus\Gamma'$, the equation (26) implies that $||x(t_1)||_2<\epsilon$ for t_1 sufficiently large. ■

By Lemma 5, it suffices to select a set of generators such that an estimate $\hat{x}_0$ can be computed, where $\hat{x}_0$ satisfies (23). The following lemma motivates a metric on C_s to achieve this requirement.

Lemma 6 ([15]): Given control input u and observation $y_s=C_s x$ for all $t\in[t_0,t_1]$, it is possible to compute an estimate $\hat{x}_0$ where the estimation error $(x_0-\hat{x}_0)$ lies in the null space of the observability Gramian

$$M(S)=\int_{t_0}^{t_1} e^{A^T(t-t_0)}C_s^T C_s e^{A(t-t_0)}\;dt. \tag{27}$$

By Lemma 6, in order to meet the estimation requirement (23), it suffices to require

$$null(M(S))\subseteq span\{v_{i\in\Gamma\setminus\Gamma'}\}. \tag{28}$$

An equivalent sufficient condition is

$$g(S)\triangleq rank\left(null(M(S))\ominus span\{v_{i\in\Gamma\setminus\Gamma'}\}\right)=0, \tag{29}$$

where $\{v_{i\in\Gamma\setminus\Gamma'}\}$ is the set of eigenvectors of stable modes.

D. Submodularity of State Observation Constraint

This section establishes the supermodular structure of $g(S)$.

Theorem 2: The set function $g(S)$ is supermodular.

Proof: Let $N = 4n$. Denote $S' = \{j \mid j \in G_k, k \in S\}$ which is a collection of indices of states that are observable at all selected generators S. We also denote $Y_{N\times N} = [y_1, \ldots, y_N]^T = e^{A(t-t_0)} \in R^{4n\times 4n}$. By approximating the integral of $M(S)$ in (27) with summation, we have

$$\begin{aligned} M(S) &= \int_{t_0}^{t_1} Y^T C_s^T C_s Y \; dt \\ &\approx \sum_{t_i=t_0}^{t_1} \sum_{k\in S} \sum_{j\in G_k} c_{jj}^2 y_j(t_i) y_j(t_i)^T \Delta t \\ &\approx \sum_{t_i=t_0}^{t_1} \sum_{k\in S'} c_{kk}^2 y_k(t_i) y_k(t_i)^T \Delta t. \end{aligned}$$

By (13) in Lemma 3, we have that

$$span\{M(S)\} = span\{y_k(t_i) \;:\; k \in S', t_i \in [t_0, t_1]\}.$$

Thus the inequality of (14) in Lemma 4 also holds for $M(S)$, i.e.,

$$\begin{aligned} &span\{M(T\cup\{v\})\} \ominus span\{M(T)\} \\ &\quad\subseteq span\{M(S\cup\{v\})\} \ominus span\{M(S)\}. \end{aligned} \tag{30}$$

Denote $\{v_{i\in\Gamma_s}\} = \{v_{i\in\Gamma\setminus\Gamma'}\}$ as the set of eigenvectors of stable modes. Since M is symmetric, we have $null(M) = R^N \ominus span(M)$, and hence

$$\begin{aligned} g(S) &= rank\left(null(M) \ominus span\{v_{i\in\Gamma_s}\}\right) \\ &= rank\left(R^N \ominus span(M) \ominus span\{v_{i\in\Gamma_s}\}\right) \\ &= rank\left(R^N \ominus \left(span\{M(S)\} \bigoplus span\{v_{i\in\Gamma_s}\}\right)\right) \\ &= N - rank\left(span\{M(S)\} \bigoplus span\{v_{i\in\Gamma_s}\}\right). \end{aligned}$$

From (30), we have

$$\begin{aligned} &(span\{M(T\cup\{v\})\} \bigoplus span\{v_{i\in\Gamma_s}\}) \\ &\qquad \ominus \; (span\{M(T)\} \bigoplus span\{v_{i\in\Gamma_s}\}) \\ &\subseteq (span\{M(S\cup\{v\})\} \bigoplus span\{v_{i\in\Gamma_s}\}) \\ &\qquad \ominus \; (span\{M(S)\} \bigoplus span\{v_{i\in\Gamma_s}\}). \end{aligned}$$

Hence

$$g(T\cup\{v\}) - g(T) \geq (S\cup\{v\}) - g(S), \tag{31}$$

which implies $g(S)$ is supermodular. ■

E. MinGen: Minimal Generator Set Selection

The problem of selecting a minimum-size set of generators is described as follows. Combining the conditions of the previous sections, we have the constraint $F(S) \triangleq f(S) + g(S) = 0$, leading to the formulation

$$\min\{|S| : F(S) = 0\} \tag{32}$$

MinGen: Algorithm for selecting generators to provide small-signal stability.

procedure MINGEN(Ω)
  $S \leftarrow \emptyset$
  while $F(S) > 0$ **do**
    for $v \in \Omega \setminus S$ **do**
      $\delta_v \leftarrow F(S \cup \{v\})$
      $v^* \leftarrow \arg\min_v \delta_v$
      $S \leftarrow S \cup \{v^*\}$
    end for
  end while
  return S
end procedure

We now present an algorithm that solves the selection problem (32).

The optimality bound of MinGen is defined by the following proposition.

Proposition 1: Let S^* denote the optimal solution to (32), and let S denote the set returned by MinGen. Then

$$\frac{|S|}{|S^*|} \leq 1 + \log\left\{\frac{m+4n}{F(S_{t-1})}\right\}, \tag{33}$$

where S_{t-1} denotes the set S at the second-to-last iteration of MinGen.

Proof: MinGen uses a greedy algorithm for generator selection. By [17], the greedy algorithm for submodular maximization of a function $\hat{f}(S)$ returns a set S satisfying

$$\frac{|S|}{|S^*|} \leq 1 + \log\left\{\frac{\hat{f}(\Omega) - \hat{f}(\emptyset)}{\hat{f}(S_t) - \hat{f}(S_{t-1})}\right\},$$

where $\hat{f}(S_t)$ is the value of the function when the algorithm terminates. In this case, $F(\Omega) - F(\emptyset) \leq m + 4n$ and $f(S_t) = 0$, completing the proof. ■

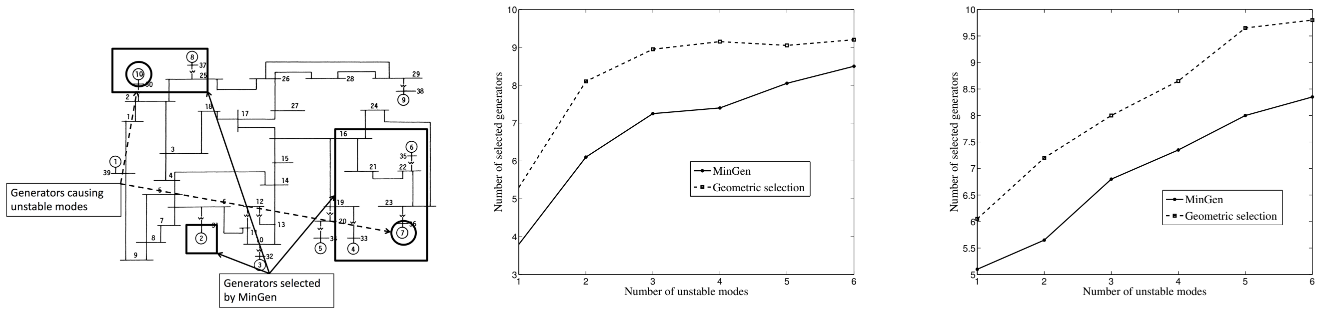
MinGen terminates in at most n iterations, while each iteration requires at most n evaluations of the objective function $F(S)$. Each evaluation of $F(S)$ solves at most $N = 4n$ least-squares problems. Solving the pseudo-inverse of $W(S)$ needs $O(N^3)$ arithmetic operations in the worst case and hence the total complexity of MinGen is $O(n^2N^4)$.

F. Robust Generator Selection Under Communication Link Failures

Communication link failures between the central controller and generators may prevent control signals from reaching the generators or measurements from reaching the controller. We model communication failures as removing columns from B and rows from C. When k communication failures occur, both matrix B and C have $\binom{n}{k}$ possible structures

$$P = \{(B_j, C_j)\}, \quad \text{for } j = 1, \ldots, \binom{n}{k} \tag{34}$$

where B_j represents the B matrix with a subset R_j of columns missing and C_j represents the C matrix with a subset R_j of row blocks missing.



(a) (b) (c)

Fig. 1: Simulation study of our proposed MinGen approach for small-signal stability. (a) IEEE 10 generator 39 bus network topology considered in this study. Circled generators have initial rotor angles differing from the rest by more than $\pi/2$. Generators selected by our algorithm are indicated by squares. (b) Comparison of average numbers of controlled generators required to meet the reachability condition (Lemma 1) using MinGen and the current state of the art. (c) Comparison of average numbers of generators for MinGen and the state of the art to achieve the observability condition (Lemma 5).

A centralized control design is robust if there exists a control $u(t)$ that is able to drive the state from x_0 to the origin in presence of k random communication failures. Defining the function $F_R(S) = F(S \setminus R)$, we observe that the stability and observability criteria are satisfied for all sets R if $\sum_{\substack{R\subseteq\Omega: \\ |R|=k}} F_R(S) = 0$. The robust selection problem can then be formulated as

$$\min_{S} \; |S| \quad s.t. \sum_{\substack{R\subseteq\Omega: \\ |R|=k}} F_R(S) = 0 \tag{35}$$

The constraint in (35) is a sum of submodular functions, and is therefore submodular. Hence, an approach analogous to MinGen can be used to select a set of generators for robust small-signal stability. The algorithm is obtained by replacing Line 5 of MinGen with $\delta_v \leftarrow \sum_{\substack{R\subseteq\Omega: \\ |R|=k}} F_R(S\cup\{v\})$. By exploiting supermodularity, optimality guarantees analogous to Proposition 1 can be obtained.

V. Performance Study of MinGen

This section presents simulation results for MinGen to damp interarea oscillations. The system topology, line reactances and generator inertia M are from the IEEE 39 bus system [8], which is also known as the 10-machine New-England Power System. Initial voltage magnitudes are set to 1.03 p.u for all generators, and initial rotor angles are varied in $[\pi/6, \pi/6+2\pi]$ till the linearized dynamics contained one or more unstable modes. The following values we obtained from generator parameters and initial operating points. The damping coefficients are $D = 0.0054M$ while AVR parameters are as follows: $T_A = 0.05I_{n\times n}$; $T_{do} = 7.7579I_{n\times n}$; $K_A = -0.2I_{n\times n}$; $K_2 = 0.1593M$; $K_3 = 5.0839T_{do}$; $K_4 = 0.0768T_{do}$; $K_5 = -64.8971T_A$; $K_6 = 865.2082T_A$. For simplicity, PSS parameters K_{pss} are set to zero.

We consider scenarios when there are up to 6 unstable modes. As shown in Figure 1(a), 2 unstable modes occur due to uncoordinated behaviors at generator 7 and 10 which have rotor angles lagging the rest by over $\pi/2$ and leading the rest by over $\pi/2$ respectively. MinGen selects the generators $\{2, 4, 6, 7, 8, 10\}$ to participate in the wide-area control.

A comparison between MinGen and the geometric indices based selection is shown in Figure 1(b) and 1(c) where each data point is an average value of 20 random trials. In each trial, the initial bus angles were chosen uniformly at random in the range $[\pi/6, \pi/6 + 2\pi]$ repeatedly until the desired number of unstable modes present. While a large number of unstable modes may not occur often in practice, we consider such scenarios for the theoretical analysis and to demonstrate the proposed algorithm's ability to correct such instability. The figures show the number of generators required for the controllability and observability conditions separately in order to compare the requirements for satisfying each condition. Our implementation of the geometric selection first computes the geometric measures of controllability $m_{ci}(k)$ and observability $m_{oi}(k)$ for each unstable mode k and each generator i as follows [9],

$$m_{ci}(k) = \frac{b_i^T\psi_k}{\|\psi_k\|\|b_i\|}, \qquad m_{oi}(k) = \frac{c_i\phi_k}{\|\phi_k\|\|c_i\|},$$

where b_i is the ith column of matrix B and c_i is the jth row of matrix C; ψ_k and ϕ_k are left and right eigenvectors of A respectively corresponding to eigenvalues λ_k. At each iteration, the generator with the highest controllability and observability indices is selected and added to the set of generators; the algorithm terminates when the conditions of Lemma 1 and Lemma 5 are satisfied. We found that the average number of generators required by wide-area control increases as the number of unstable modes increases. We also observed that MinGen requires fewer generators for both receiving control signals and sending observations.

VI. Discussion and Conclusions

This paper studied the problem of selecting a set of generators for wide-area control in order to ensure small-signal stability of the power system, which is one of the critical stability challenges identified by NASPI [4]. We introduced

MinGen, a framework for selecting generators to receive control inputs and send state information to the controller. We formulated the problem of selecting the minimum-size set of generators to guarantee that the distance from the unstable modes to the span of the controllability Gramian is zero. In order to guarantee that the set of generators is sufficient to compute the stabilizing control signal, we formulated the problem of ensuring that the null space of the observability Gramian is contained in the span of the stable modes. We showed that both of these discrete optimization problems are submodular as a function of the set of generators, and developed efficient algorithms with provable optimality bounds for selecting generators to ensure small-signal stability. We then proposed algorithms for generator selection under communication link failures. Our results were validated through numerical study on the IEEE New England Test Case. Our future work will include joint design of the set of generators and the control action in order to minimize the total control effort.

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APPENDIX

Proof of Lemma 3: We first show the equality in (13) holds when $\{w_k, k \in S\}$ are linearly independent. Then we generalize the result to the case when $\{w_k, k \in S\}$ may be linearly dependent. Suppose without of generality that $S = \{1, \ldots, l\}$ and denote $X = [w_1, \ldots, w_l] \in R^{N \times l}$.

Case 1: When $\{w_{k\in S}\}$ are linearly independent, for any point $p \in span\{\sum_{k\in S} w_k w_k^T\}$, $\exists v \in R^N$ and $z \in R^l$ such that $X^T v = z$. Multiplying X on both sides gives $p = XX^T v = Xz$ which implies $p \in span\{w_{k\in S}\}$ and hence

$$span\{\sum_{k\in S} w_k w_k^T\} \subseteq span\{w_{k\in S}\}. \tag{36}$$

On the other hand, for any point $q \in span\{w_{k\in S}\}$, $\exists z \in R^l$ and $v \in R^N$ such that $z = X^T v$ as $\{w_{k\in S}\}$ are linearly independent. Thus we have $q = Xz = XX^T v$ which implies $q \in span\{\sum_{k\in S} w_k w_k^T\}$ and hence

$$span\{\sum_{k\in S} w_k w_k^T\} \supseteq span\{w_{k\in S}\}. \tag{37}$$

Results (36) and (37) give the equality of (13).

Case 2: When w_v is linearly dependent of $\{w_{k\in S}\}$, there exist constants $\{\gamma_1, \ldots, \gamma_l\}$ such that $w_v = \sum_{k\in S} \gamma_k w_k$. Then we have

$$\begin{aligned}\sum_{k\in S} w_k w_k^T + w_v w_v^T =& \sum_{k\in S} w_k w_k^T + (\sum_{k\in S} \gamma_k w_k)(\sum_{k\in S} \gamma_k w_k^T)\\ =& X(V+I)X^T\end{aligned}$$

where I is identity matrix with size $l \times l$ and

$$V = \begin{bmatrix} 1+\gamma_1^2 & \gamma_1\gamma_2 & \cdots & \gamma_1\gamma_l \\ \gamma_1\gamma_2 & 1+\gamma_2^2 & \cdots & \gamma_2\gamma_l \\ \vdots & & \ddots & \vdots \\ \gamma_1\gamma_l & \gamma_2\gamma_l & \cdots & 1+\gamma_l^2 \end{bmatrix}.$$

Note that $(V + I)$ is positive definite and hence invertible.

For any element $p = XX^T v \in span\{\sum_{k\in S} w_k w_k^T\}$, there always exists a vector $z \in R^N$ such that $(V + I)^{-1} X^T v = X^T z$, as columns of X are linearly independent. Therefore we have $p = XX^T v = X(V+I)X^T z$ which implies $p \in span\{\sum_{k\in S} w_k w_k^T + w_v w_v^T\}$.

The process above is invertible: given any $z \in R^N$, a $v \in R^l$ can be found such that $(V + I)X^T z = X^T v$ and hence $X(V + I)X^T z = XX^T v$.

Therefore, when w_v is linearly dependent of $\{w_{k\in S}\}$,

$$span\{\sum_{k\in S} w_k w_k^T + w_v w_v^T\} = span\{\sum_{k\in S} w_k w_k^T\}. \tag{38}$$

By the result in case 1, $span\{\sum_{k\in S} w_k w_k^T\} = span\{w_{k\in S}\} = span\{w_{k\in S}, w_v\}$, and hence we have

$$span\{\sum_{k\in S} w_k w_k^T + w_v w_v^T\} = span\{w_{k\in S}, w_v\}, \tag{39}$$

which completes the proof. ■