EE485 Introduction to Photonics

Diffraction
1. Fraunhofer diffraction
2. Diffraction grating
3. Fresnel diffraction

Reading: Pedrotti, Chapter 11, Sec. 12.1-12.5, 13.1-13.2, 13.4-13.6
Diffraction: Deviation from geometrical optics that results from the obstruction of a wavefront of light.

Diffraction patterns through apertures with various shapes

Underlying principle: Interference
Young’s Double-slit Experiment Revisited

The slit is considered as having no width.

Diffraction: The slit has finite width.
Fraunhofer vs. Fresnel Diffraction

**Fraunhofer diffraction**  
*(Far-field diffraction):* Both the light source and the observation plane are far away from the aperture. The wavefronts arriving at the aperture and observation screen may be considered planar.

**Fresnel diffraction**  
*(Near-field diffraction):* Both the light source and the observation plane are close to the aperture. The wavefronts arriving at the aperture and observation screen are not planar. The curvature of the wavefronts must be taken into account.
Fraunhofer Diffraction from a Single Slit

Partition the slit into intervals of dimension $ds$.

$$dE_p = \left( \frac{E_L ds}{r_0} \right) e^{i(kr_0 + ks \sin \theta - \omega t)}$$

$E_L$: Amplitude per unit width

$$E_p = \left( \frac{E_L}{r_0} \int_{-b/2}^{b/2} e^{i k s \sin \theta} ds \right) e^{i(kr_0 - \omega t)}$$

$$E_p = \frac{E_L}{r_0} e^{i(kr_0 - \omega t)} \left( \frac{e^{i k s \sin \theta}}{i k \sin \theta} \right)_{-b/2}^{b/2}$$

$$\beta \equiv \frac{1}{2} kb \sin \theta$$

$\beta$ indicates where the observation point is.

Intensity at point $P$:

$$I = \frac{|E_p|^2}{2\eta_0} = \frac{\varepsilon_0 c}{2} \left( \frac{E_L b}{r_0} \right)^2 \frac{\sin^2 \beta}{\beta^2}$$
### Single-slit Diffraction Pattern

\[ I = I_0 \frac{\sin^2 \beta}{\beta^2} = I_0 \text{sinc}^2 \beta \]

**Main peak:** \( \beta = 0 \) (\( \theta = 0 \)), \( \text{sinc}^2 \beta = 1 \)

**Zeros:** \( \beta = \frac{1}{2} kb \sin \theta = m\pi, \quad m = \pm 1, \pm 2, \ldots \)

\[ \rightarrow m\lambda = b \sin \theta \]

Where are the secondary maxima?

\[ \frac{d}{d\beta} \left( \frac{\sin \beta}{\beta} \right) = \frac{\beta \cos \beta - \sin \beta}{\beta^2} = 0 \]

\[ \rightarrow \beta = \tan \beta \]
Beam Spreading

First zeros: \( b \sin \theta = \pm \lambda \rightarrow \Delta \theta = \frac{2\lambda}{b} \)

Example: Imagine a parallel beam of 650 nm light with width 0.5 mm, propagating across a distance of 10 m. Determine the final width of the beam. What if the beam passes through a 0.1-mm slit aperture at the beginning?

How far is far-field?

The beam width cannot be smaller than the aperture. \( \rightarrow L_{\text{min}} (W = b) = \frac{b^2}{2\lambda} \)

Far field: \( L \gg \frac{b^2}{\lambda} \)  
In general, \( L \gg \frac{\text{area of aperture}}{\lambda} \)
Diffraction from Rectangular Apertures

\[ I = I_0 (\text{sinc}^2 \beta)(\text{sinc}^2 \alpha) \]

\[ \alpha = \frac{1}{2} ka \sin \theta \]
Diffraction from Circular Apertures

\[ E_p = \frac{E_A}{r_0} e^{i(kr_0 - \omega t)} \iint_{\text{Area}} e^{iks \sin \theta} dA \]

\[ E_A : \text{Amplitude per unit area} \]

\[ \begin{align*}
E_p &= \frac{2E_A}{r_0} e^{i(kr_0 - \omega t)} \int_{-R}^{R} e^{iks \sin \theta} \sqrt{R^2 - s^2} \, ds \\
&= \frac{2E_A R^2}{r_0} e^{i(kr_0 - \omega t)} \int_{-1}^{1} e^{i\gamma} \sqrt{1 - \nu^2} \, d\nu \\
&= \frac{2E_A R^2}{r_0} e^{i(kr_0 - \omega t)} \frac{\pi J_1(\gamma)}{\gamma}
\end{align*} \]

\[ J_1(\gamma) : \text{1st-order Bessel function of the 1st kind} \]

\[ J_1(\gamma) = \frac{\gamma}{2} - \frac{(\gamma/2)^3}{1^2 \cdot 2} + \frac{(\gamma/2)^5}{1^2 \cdot 2^2 \cdot 3} - \ldots \]

\[ \frac{J_1(\gamma)}{\gamma} \to \frac{1}{2} \quad \text{as} \quad \gamma \to 0 \]
Diffraction Pattern from a Circular Aperture

\[ I = I_0 \left( \frac{2J_1(\gamma)}{\gamma} \right)^2, \quad \gamma = \frac{kD}{2} \sin \theta \quad (D: \text{diameter of the aperture}) \]

First zero: \[ \gamma = \frac{kD}{2} \sin \theta = 3.832 \rightarrow D \sin \theta = 1.22\lambda \]

Far-field angular radius: \[ \Delta \theta_{1/2} = \frac{1.22\lambda}{D} \]
Resolution of an Imaging Instrument

One source or two sources?

Rayleigh’s criterion for just-resolvable images

\[ (\Delta \theta)_{\text{min}} = \frac{1.22\lambda}{D} \]

\( D \): Diameter of the lens
Example: Resolution of a Microscope

For typical microscope objectives, N.A. ~ 0.5.

Improve the resolution by oil-immersion lens (Sec. 3-6):

For a good oil-immersion objective, N.A. ~ 1.2

\[ x_{\text{min}} = f \theta_{\text{min}} = f \left( \frac{1.22 \lambda}{D} \right) \]

\[ \frac{D}{f} : \text{Numerical aperture (N.A.)} \]

\[ x_{\text{min}} \approx \lambda \]
Double-slit Diffraction

$$E_R = \frac{E_L}{r_0} \int_{-(1/2)(a-b)}^{-(1/2)(a+b)} e^{ik_0\sin \theta} \, ds + \frac{E_L}{r_0} \int_{(1/2)(a-b)}^{(1/2)(a+b)} e^{ik_0\sin \theta} \, ds$$

$$\beta = \frac{1}{2} kb \sin \theta, \quad \alpha = \frac{1}{2} ka \sin \theta$$

$$E_R = \frac{2E_L}{r_0} \frac{\sin \beta}{\beta} \cos \alpha$$

$$I = \left( \frac{\varepsilon_0 c}{2} \right) \left( \frac{E_R}{r_0} \right)^2 = 4I_0 \left( \frac{\sin \beta}{\beta} \right)^2 \cos^2 \alpha$$

$$I_0 = \left( \frac{\varepsilon_0 c}{2} \right) \left( \frac{E_L b}{r_0} \right)^2 \quad \text{(maximum in single-slit diffraction)}$$

Some features:

$$\left( \frac{\sin \beta}{\beta} \right)^2 \rightarrow \text{Single-slit diffraction}$$

$$\cos^2 \alpha = \cos^2 \left[ \frac{ka(\sin \theta)}{2} \right] = \cos^2 \left[ \frac{\pi a(\sin \theta)}{\lambda} \right]$$

$$\rightarrow \text{Double-slit interference}$$

Maximum intensity is four times the maximum in single-slit diffraction.
Double-slit vs. Single-slit diffraction

Interference

Single-slit diffraction

Double-slit diffraction
Diffraction from Many Slits

\[ E_R = \frac{E_L}{r_0} \frac{N/2}{\sum_{j=1} \left\{ \int_{[-(2j-1)a+b]/2}^{[(2j-1)a+b]/2} e^{i k s \sin \theta} ds + \int_{[(2j-1)a-b]/2}^{[(2j-1)a+b]/2} e^{i k s \sin \theta} ds \right\} \sin \beta \sin N \alpha}{\sin \alpha} \]

\[ \beta \equiv \frac{1}{2} k b \sin \theta, \quad \alpha \equiv \frac{1}{2} k a \sin \theta \]

\[ E_R = \frac{E_L b}{r_0} \frac{\sin \beta}{\beta} \frac{\sin N \alpha}{\sin \alpha} \]

\[ I = \left( \frac{\varepsilon_0 c}{2} \right) E_R^2 = I_0 \left( \frac{\sin \beta}{\beta} \right)^2 \left( \frac{\sin N \alpha}{\sin \alpha} \right)^2 \]

\[ I_0 = \left( \frac{\varepsilon_0 c}{2} \right) \left( \frac{E_L b}{r_0} \right)^2 \text{ (maximum in single-slit diffraction)} \]

Compare this with multi-wave interference,

\[ I = I_0 \frac{\sin^2(M \phi/2)}{\sin^2(\phi/2)} \Rightarrow \frac{\phi}{2} \leftrightarrow \alpha = \frac{1}{2} k a \sin \theta \]

L'Hopital's rule for \( m = 0, \pm 1, \pm 2, \ldots \)

\[ \lim_{\alpha \to m \pi} \frac{\sin N \alpha}{\sin \alpha} = \lim_{\alpha \to m \pi} \frac{N \cos N \alpha}{\cos \alpha} = \pm N \]

What is this term?

Single-slit interference

(Multi-wave interference)
Multi-slit Diffraction Pattern

Let’s guess how many slits produce these interference patterns …

N = 4

N = 5
**Diffraction Grating**

Normal incidence

\[ \alpha = \frac{1}{2} ka \sin \theta = m\pi \rightarrow m\lambda = a \sin \theta \]

Oblique incidence

\[ m\lambda = a(\sin \theta_i + \sin \theta_m) \]

\( \theta_m \) is positive when the incident and diffracted rays are on the same side of the normal.
Dispersion of a Grating

Separate light with different wavelengths by a grating.

Angular dispersion

\[ \mathcal{D} \equiv \frac{d\theta_m}{d\lambda} = \frac{m}{a \cos \theta_m} \]

At normal incidence,

\[ \mathcal{D} = \frac{\tan \theta_m}{\lambda} \]

Example: Light of wavelength 500 nm is incident normally on a grating with 5000 grooves/cm. Determine the diffraction angle and angular dispersion for 1st and 2nd order diffraction.

Demonstration: Diffraction from a CD.
Resolution of a Grating

\[ a \sin \theta = m\lambda + d\lambda \]

\[ a \sin \theta = \left( m + \frac{1}{N} \right) \lambda \]

Resolving power

\[ R \equiv \frac{\lambda}{(\Delta \lambda)_{\text{min}}} = mN = \frac{W \sin \theta_m}{\lambda} \]

\( W \): width of the grating covered by the light.

Example: A grating of 5000 grooves/cm has a width of 8 cm fully covered by the incident light. What’s the smallest wavelength difference that can be resolved around \( \lambda = 500 \) nm by 1\textsuperscript{st} order and 2\textsuperscript{nd} order diffraction, respectively?
Fresnel Diffraction

\[ E_P = \frac{-ikE_S e^{-i\omega t}}{2\pi} \iint F(\theta) e^{ik(r+r')} \frac{dA}{rr'} \]

\[ F(\theta) = \frac{1 + \cos \theta}{2} \]

**Criterion:**

\[ \Delta = p - \sqrt{r'^2 - h^2} \approx p - r' \left(1 - \frac{h^2}{2r'^2}\right) \]

Near-field \(\rightarrow\) Significant curvature

\[ \Delta \approx \frac{h^2}{2p} > \lambda, \quad \Delta \approx \frac{h^2}{2q} > \lambda \]

\[ \rightarrow \frac{1}{2} \left(\frac{1}{p} + \frac{1}{q}\right) h^2 > \lambda \]

In general,

\[ d < \frac{A}{\lambda} \]

\(d:\) \(p \) or \(q\), \(A\): Area of the aperture
Fresnel Diffraction from Circular Apertures

- Dividing the aperture into Fresnel zones with circular symmetry.
- Each zone is, on the average, $\lambda/2$ farther from the field point $P$ than the preceding zone.

$$r_1 = r_0 + \lambda/2, \quad r_2 = r_0 + \lambda, \ldots, \quad r_N = r_0 + N\lambda/2$$

Summation of the waves at $P$ from each half-period zone:

$$A_n = a_1 + a_2 e^{i\pi} + a_3 e^{i2\pi} + a_4 e^{i3\pi} + \cdots a_n e^{i(n-1)\pi}$$

$$= a_1 - a_2 + a_3 - a_4 + \cdots a_n \approx a_1 / 2 \text{ for large } n$$

Quiz: What’s the diffraction pattern from an opaque, circular disc covering the 1$^{st}$ Fresnel zone?

Amplitude $\propto a_2 / 2$
The Fresnel Zone Plate

Blocking the zones that contribute to the negative (or positive) terms.

\[
R_n^2 = \left( r_0 + \frac{n\lambda}{2} \right)^2 - r_0^2
\]

\[
R_n \approx \sqrt{n r_0 \lambda} \quad \text{for } n\lambda / r_0 \ll 1
\]

The zone plate operates as a lens with P as the primary focal point.

\[
f_1 = \frac{R_1^2}{\lambda}
\]

\[
f_n = \frac{R_1^2}{n\lambda}, \quad n \text{ odd}
\]
Examples of Fresnel Lens

Collimation of divergent beam by the micromachined Fresnel zone plate.

The big one: Ponce De Leon Inlet Lighthouse near Daytona Beach Florida

The small one: Micromachined Fresnel zone plate

FWHM Beam Width (μm)

Without lens (θ_{FWHM}=5.0°)
Collimated by lens (θ_{FWHM}=0.33°)

Distance From Fiber (cm)

100 μm
(a) What are the beam profiles of a semiconductor laser at near-field and far-field? (b) Design a Fresnel lens to collimate this laser beam with a specific focal length. What are the shapes of the zones?