EE 485 Homework #1 Solutions

1. (20%) A light wave propagating in a medium of refractive index $n$ can be expressed as $y = A[1 - 2\sin^2(10^7 x - 2\times10^{15} t)]$, with $x$ in meters and $t$ in seconds. We know that this light wave is a harmonic wave. Determine the following properties of the light wave: (a) propagation constant, (b) angular frequency, (c) wavelength in the medium, (d) frequency, (e) period, and (f) velocity in the medium and propagation direction. Also, determine (g) the refractive index of the medium and (h) wavelength of the light wave in free space.

\[
\cos 2\theta = 1 - 2\sin^2 \theta
\]
\[
\therefore y = A \cos(2\times10^7 x - 4\times10^{15} t) = A \cos(kx - \omega t)
\]

(a) $k = 2\times10^7$ \(\text{(1/m)}\)
(b) $\omega = 4\times10^{15}$ \(\text{(rad/s)}\)
(c) $\lambda = \frac{2\pi}{k} = \frac{2\pi}{2\times10^7} = 3.14\times10^{-7}$ \(\text{(m)}\) = 314 \(\text{(nm)}\)
(d) $f = \frac{\omega}{2\pi} = \frac{4\times10^{15}}{2\pi} = 6.37\times10^{14}$ \(\text{(Hz)}\)
(e) $T = \frac{1}{f} = \frac{1}{6.37\times10^{14}} = 1.57\times10^{-15}$ \(\text{(s)}\) = 1.57 \(\text{(fs)}\)
(f) $v = \lambda f = 3.14\times10^{-7} \times 6.37\times10^{14} = 2\times10^8$ \(\text{(m/s)}\), in +x-direction.
   (Note: You can also determine this from the wave function directly.)

(g) $n = \frac{c_0}{v} = \frac{3\times10^8}{2\times10^8} = 1.5$
(h) $\lambda_0 = n\lambda = 1.5\times314 = 471$ \(\text{(nm)}\)

2. (20%) (Pedrotti 4-11) By finding appropriate expressions for $\vec{k} \cdot \vec{r}$, write equations describing a sinusoidal plane wave in three dimensions, displaying wavelength and velocity, if propagation is
   (a) along the +z-axis
   (b) along the line $x = y, z = 0$ (two possible propagation directions)
   (c) perpendicular to the planes $x + y + z = \text{constant}$ (two possible propagation directions)

(a) For propagation along the z-axis, $k_x = k_y = 0$. So $\vec{k} \cdot \vec{r} = k_z \cdot z$, with $k_z = 2\pi / \lambda$. The waveform can then be written as
   \[
   \psi = A \sin(k_z \cdot z - \omega t) = A \sin \left[ \frac{2\pi}{\lambda} (z - vt) \right]
   \]

(b) In this case, $k_z = 0$ and $k_x = k_y = |k| / \sqrt{2} = \frac{2\pi}{\sqrt{2}\lambda}$. The general form of the wave is then
   \[
   \psi = A \sin(k_x \cdot x + k_y \cdot y + \omega t) = A \sin \left[ \frac{2\pi}{\sqrt{2}\lambda} (x + y \pm \sqrt{2}vt) \right]
   \]
(c) In this case, \( \vec{k} = \frac{k}{\sqrt{3}} (\hat{x} + \hat{y} + \hat{z}) \) and \( \vec{k} \cdot \vec{r} = \frac{k}{\sqrt{3}} (x + y + z) \), with \( k = 2\pi / \lambda \). The waveform is then
\[
\psi = A \sin \left[ \frac{k}{\sqrt{3}} (x + y + z) \pm \omega t \right] = A \sin \left[ \frac{2\pi}{\sqrt{3} \lambda} (x + y + z \pm \sqrt{3} vt) \right]
\]

3. (20%) Traveling standing wave
(For our classroom discussion on Monday 1/12/2015, please work on (1) the mathematic part of (a), that is, applying the electrical field to \( \nabla^2 E_x + k^2 E_x = 0 \), \( \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \). (2) The 1st part of (d), expressing \( \mathbf{E}(\mathbf{r}) = E_0 \cos \beta y \exp(i\beta z) \mathbf{x} \) as the sum of two plane waves in the form of \( e^{ik\cdot\mathbf{r}} \) by expressing \( \cos \beta y = \frac{1}{2} \left( e^{i\beta y} + e^{-i\beta y} \right) \). We will discuss the physical meaning of this wave in class.)

The complex wave function of the electric field of a monochromatic light wave has a space component of \( E(r) = E_0 \cos \beta y \exp(i\beta z) \mathbf{x} \). Its wavelength is \( \lambda_0 \). (a) Use the Helmholtz equation, determine the relation between \( \beta \) and \( \lambda_0 \) (Note: Remember that \( k = \frac{2\pi}{\lambda_0} \). You will find out that \( \beta \neq k \) in this case). (b) Derive an expression for the space component of the magnetic field \( \mathbf{H}(\mathbf{r}) \). (c) Determine the direction of flow of the average optical power. (Hint: Use Poynting vector.) (d) This wave may be regarded as the sum of two TEM plane waves. Express the electric field and magnetic field as the sum of two plane waves. Determine their directions of propagation. Explain why this is a “traveling standing wave.”

(a) Each component in the electric field needs to satisfy the Helmholtz equation,
\[
\nabla^2 E_x + k^2 E_x = 0
\]
\[
\rightarrow -2\beta^2 E_0 \cos \beta y \exp(i\beta z) + k^2 E_0 \cos \beta y \exp(i\beta z) = 0
\]
\[
\rightarrow \beta = \frac{1}{\sqrt{2}} k = \frac{1}{\sqrt{2}} \frac{2\pi}{\lambda_0}
\]
(b) \( \vec{H}(\vec{r}) = \frac{1}{i\omega \mu_0} \nabla \times \vec{E}(\vec{r}) = \frac{1}{i\omega \mu_0} \left( \hat{y} \frac{\partial E_x}{\partial z} - \hat{z} \frac{\partial E_x}{\partial y} \right) \)
\[
\vec{H}(\vec{r}) = \frac{E_0 \exp(i\beta z)}{\sqrt{2} \mu_0 c_0} \left( \cos \beta y \hat{y} - i \sin \beta y \hat{z} \right)
\]
(c) Average optical power flow with direction =
\[
\text{Re}\left\{ \frac{1}{2} \mathbf{E}(\mathbf{r}) \times \mathbf{H}(\mathbf{r})^* \right\} = \text{Re}\left\{ \frac{1}{2} E_0 \cos \beta y \exp(i\beta z) \hat{x} \times \frac{E_0 \exp(-i\beta z)}{\sqrt{2\mu_0 c_0}} \left( \cos \beta \hat{y} + i \sin \beta \hat{z} \right) \right\} = \frac{E_0^2}{2\sqrt{2\mu_0 c_0}} \cos^2 \beta y \hat{z}
\]

The optical power flows along +z-direction.

(d) \( \mathbf{E}(\mathbf{r}) = E_0 \cos \beta y \exp(i\beta z) \hat{x} = E_0 \left[ e^{i(\beta y + \xi z)} + e^{-i(\beta y - \xi z)} \right] \hat{x} = \frac{E_0}{2} \left( e^{i\xi z} + e^{-i\xi z} \right) \hat{x} \)

\[
\mathbf{H}(\mathbf{r}) = \frac{E_0 \exp(i\beta z)}{\sqrt{2\mu_0 c_0}} \left( \cos \beta \hat{y} - i \sin \beta \hat{z} \right) = \frac{E_0}{2\sqrt{2\mu_0 c_0}} \left[ e^{i(\beta y + \xi z)} + e^{-i(\beta y - \xi z)} \right] \hat{y} - \frac{E_0}{2\sqrt{2\mu_0 c_0}} \left[ e^{i(\beta y + \xi z)} - e^{-i(\beta y - \xi z)} \right] \hat{z}
\]

This wave is a sum of two plane wave. The first one propagates in \( \mathbf{k}_1 = \frac{k}{\sqrt{2}} (\hat{y} + \hat{z}) \) direction, with \( \mathbf{E}(\mathbf{r}) = \frac{E_0}{2} \left( e^{i\xi z} \right) \hat{x} \) and \( \mathbf{H}(\mathbf{r}) = \frac{E_0}{2\sqrt{2\mu_0 c_0}} (\hat{y} - \hat{z}) e^{i\xi z} \hat{x} \). The second one propagates in \( \mathbf{k}_2 = \frac{k}{\sqrt{2}} (-\hat{y} + \hat{z}) \) direction, with \( \mathbf{E}(\mathbf{r}) = \frac{E_0}{2} \left( e^{i\xi z} \right) \hat{x} \) and \( \mathbf{H}(\mathbf{r}) = \frac{E_0}{2\sqrt{2\mu_0 c_0}} (\hat{y} + \hat{z}) e^{i\xi z} \hat{x} \). The net result is a traveling wave in +z-direction and a standing wave pattern \( \cos \beta y \) along the y-direction.

4. (20%) The solar constant is the radiant flux density (irradiance or intensity) from the sun at the surface of the earth’s atmosphere and is about 0.135 W/cm\(^2\). Assume an average wavelength of 700 nm for the sun’s radiation that reaches the earth. (a) Find the amplitudes of the electric field \( E_0 \) (V/m) and the magnetic field \( H_0 \) (A/m). (b) The sun light is absorbed by a semiconductor photovoltaic cell. Assume the susceptibility \( \chi \) of the semiconductor is 11.25, determine its refractive index \( n \) and impedance \( \eta \). (c) If the average power attenuation coefficient \( \alpha \) of the sun light due to absorption by the photovoltaic cell is 1.2 cm\(^{-1}\), determine the propagation constant \( \beta \) of the sun light in the semiconductor and the complex propagation constant \( k \). After propagating in the semiconductor for 1 mm, what will the optical intensity be? (d) Write down the complex wave functions \( \mathbf{E}(\mathbf{r}, t) \) of the sun light in the
semiconductor, using the average wavelength as \( \lambda \) and assuming the propagation direction is +z-direction. Remember to include the attenuation and assume no reflection loss.

(a) \( E_0 = \sqrt{2 \eta_0 I_0} = \sqrt{2 \times 377 \times 0.135 \times 10^4} = 1009 \) (V/m)

\[
H_0 = \frac{E_0}{\eta_0} = \frac{1009}{377} = 2.7 \text{ (A/m)}
\]

(b) \( n = \sqrt{1 + \chi} = \sqrt{1 + 11.25} = 3.5 \)

\[
\eta = \frac{n \eta_0}{n} = \frac{377}{3.5} = 107.7 \text{ (\Omega)}
\]

(c) \( \beta = \frac{2 \pi n}{\lambda_0} = \frac{2 \pi \times 3.5}{700 \times 10^{-9}} = 3.14 \times 10^7 \text{ (1/m)} \)

\( \alpha = 1.2 \text{ cm}^{-1} = 120 \text{ m}^{-1} \)

\( \therefore k = \beta + \frac{i}{2} \alpha = 3.14 \times 10^7 + 60i \text{ (1/m)} \), \( \vec{k} = k \hat{z} \)

\[
I(1 \text{ mm}) = I_0 e^{-\alpha \lambda} = 0.135 \cdot e^{-1.2 \times 0.1} = 0.12 \text{ W/cm}^2
\]

(d) \( \omega = \frac{2 \pi c_0}{\lambda_0} = \frac{2 \pi \times 3 \times 10^8}{700 \times 10^{-9}} = 2.7 \times 10^{15} \text{ (rad/s)} \)

\[
E(\vec{r}, t) = E_0 e^{i(kz - \omega t)} = 1009 \exp[i(3.14 \times 10^7 z - 2.7 \times 10^{15} t)] \exp(-60z) \text{ (V/m)}
\]

5. (20%) A step-index fiber 50 \( \mu \)m in core diameter has a core index of 1.50 and a cladding index of 1.45. Determine (a) the numerical aperture of the fiber; (b) the acceptance angle; (c) the number of reflections in 1 meter of fiber for a ray at the maximum entrance angle, and for one at half this angle. If the transmission loss is mostly caused by the roughness at the core/cladding interface and is proportional to the number of reflections, which ray will experience higher transmission loss? Assume \( \lambda = 1310 \) nm, (d) how many modes exist in this fiber? (e) What is the maximum core diameter if we want to make this into a single-mode fiber?

(a) \( N.A. = \sqrt{n_{core}^2 - n_{cladding}^2} = \sqrt{1.50^2 - 1.45^2} = 0.384 \)

(b) \( \theta_m = \sin^{-1}(0.384) = 22.6^\circ \)

(c) Number of reflections per meter for \( \theta_m = \theta_m \):

\[
\tan(\sin^{-1}(\sin \theta_m / 1.50)) \times 50 \times 10^{-6} = 5296 \text{ / meter}
\]

Number of reflections per meter for \( \theta_m = \frac{1}{2} \theta_m \):

\[
\tan(\sin^{-1}(\sin(1/2 \theta_m) / 1.50)) \times 50 \times 10^{-6} = 2635 \text{ / meter}
\]

The ray incident at the maximum entrance angle will experience higher transmission loss.
(d) \[ m_{\text{max}} \approx \frac{1}{2} \left( \frac{\pi d}{\lambda N.A.} \right)^2 = \frac{1}{2} \left( \frac{\pi \cdot 50}{1.31} \right)^2 = 1060 \] (an answer of 1061 is correct too)

(e) \[ d < \frac{2.4}{\pi (N.A.)} \cdot \lambda = \frac{2.4}{\pi \cdot 0.384} \cdot 1.31 = 2.61 \text{ \mu m} \]

Note that by choosing too big a difference between core index and cladding index (large N.A.), a serious restriction is put on the diameter of the single mode fiber!