EE485 Homework #4 Solutions

1. (15%) P.11-3 of Pedrotti. Measure slit width by diffraction.
   The width of a rectangular slit is measured in the laboratory by means of its diffraction pattern at a distance of 2 m from the slit. When illuminated normally with a parallel beam of laser light (632.8 nm), the distance between the third minima on either side of the principal maximum is measured. An average of several tries gives 5.625 cm.
   (a) Assume Fraunhofer diffraction, what is the slit width?
   (b) Is the assumption of far-field diffraction justified in this case? What is the ratio $L/L_{\text{min}}$?

   (a) Minima: $m\lambda = b\sin \theta = b \cdot \frac{\gamma}{L}$
   
   $y_{+3} - y_{-3} = \Delta y = (3 - (-3))\lambda \cdot \frac{L}{b} \rightarrow b = \frac{6\lambda L}{\Delta y} = \frac{6(632.8 \text{ nm})(2 \text{ m})}{5.625 \text{ cm}} = 0.0135 \text{ cm} = 0.135 \text{ mm}$

   (b) $L_{\text{min}} = \frac{b^2}{2\lambda} = \frac{(0.0135 \text{ cm})^2}{2 \cdot (632.8 \text{ nm})} = 1.44 \text{ cm}$

   $\therefore L = 200 \text{ cm} \gg L_{\text{min}}, \frac{L}{L_{\text{min}}} = 139$

2. (15%) P.12-2 of Pedrotti. Dispersion and resolving power of a grating.
   (a) Describe the dispersion in the red wavelength region around 650 nm (both in $^\circ$/nm and in mm/nm (Note: this is slightly different from what the textbook asks for.)) for a transmission grating 6 cm wide, containing 3500 grooves/cm, when it is focused in the third-order spectrum on a screen by a lens of focal length 150 cm.
   (b) Find the resolving power of the grating under these conditions.

   (a) For normal incidence, $m\lambda = a\sin \theta_m$

   \[ D = \frac{d\theta_m}{d\lambda} = \frac{m}{a\cos \theta_m} = \frac{a}{a\cos(\sin^{-1} m\lambda/a)} \]

   \[ D = \frac{3}{3500} \cos \left( \sin^{-1} \frac{3\times650\times10^{-7}}{1/3500} \right) = 14366 \text{ rad/cm} \]

   In $^\circ$/nm, \[ D = 14366 \text{ rad/cm} \times \frac{180 \text{ deg}}{\pi \text{ rad}} \times \frac{\text{cm}}{10^{-7} \text{ cm}} = 0.0823 \text{ deg/nm} \]

   In mm/nm, \[ \frac{dy}{d\lambda} = f \frac{d\theta}{d\lambda} = fD = (1500 \text{ mm}) \times (14366 \times 10^{-7} \text{ rad/nm}) = 2.1549 \text{ mm/nm} \]

   (b) \[ \mathcal{R} = mN = 3 \times 3500 \times 6 = 63,000 \]

3. (20%) P.12-3 of Pedrotti. Angular separation and resolving power of a grating.
   Assume normal incidence.
   (a) What is the angular separation between the 2nd-order principal maximum and the neighboring minimum on either side for the Fraunhofer pattern of a 24-groove grating having a groove separation of $10^{-3}$ cm and illuminated by light of 600 nm?
   (b) What slightly longer (or slightly shorter) wavelength would have its 2nd-order maximum on top of the minimum adjacent to the 2nd-order maximum of 600-nm light?
(c) From your results in parts (a) and (b), calculate the resolving power in 2\textsuperscript{nd} order. Compare this with the resolving power obtained from the theoretical grating resolving power formula, Eq. (12-11).

(Note: Assume normal incidence.)

(a) Principle maxima: \( m\lambda = a \sin \theta_m \)

\[
\theta_2 = \sin^{-1}\left(\frac{2 \times 600 \times 10^{-7}}{10^{-3}}\right) = 6.892^\circ
\]

Minima: \( \left(m \pm \frac{1}{N}\right)\lambda = a \sin \theta \)

\[
\theta_{\min,2} = \sin^{-1}\left(\frac{m \pm \frac{1}{N}}{a}\right) \lambda = \sin^{-1}\left(\frac{2 \pm \frac{1}{24}}{10^{-3}}\right)(600 \times 10^{-7}) = 7.0364^\circ \text{ for "+" and } 6.7478^\circ \text{ for "-".}
\]

\[\Delta \theta = |\theta_2 - \theta_{\min,2}| = 0.1443^\circ \text{ for both cases.}\]

(b) For "+" case, \( \lambda = \frac{a \sin \theta_m}{m} = \frac{(10^4 \text{ nm}) \sin 7.0364^\circ}{2} = 612.5 \text{ nm} \)

For "-" case, \( \lambda = \frac{a \sin \theta_m}{m} = \frac{(10^4 \text{ nm}) \sin 6.7478^\circ}{2} = 587.5 \text{ nm} \)

(c) \( (\Delta \lambda)_{\min} = (612.5 \text{ nm}) - (600 \text{ nm}) \text{ or } (600 \text{ nm}) - (587.5 \text{ nm}) = 12.5 \text{ nm} \)

\[
R_{m=2} = \frac{\lambda}{(\Delta \lambda)_{\min}} = \frac{600}{12.5} = 48
\]

Using Eq. (12-11), \( R = mN = 2 \times 24 = 48 \). The results are the same.

4. (15%) P.12-14 of Pedrotti\textsuperscript{3}. Blazed grating.

A reflection grating, ruled over a 15-cm width, is to be blazed for use at 200 nm in the vacuum ultraviolet. If its theoretical resolving power in 1\textsuperscript{st} order is to be 300,000, determine the proper blaze angle for use (a) in a Littrow mount and (b) with normal incidence.

\[
R = mN = \frac{W}{a} \Rightarrow a = \frac{mW}{R} = \frac{15}{300,000} = 5 \times 10^{-5} \text{ cm} = 500 \text{ nm}
\]

(a) Littrow mount: \( \theta_b = \sin^{-1}\left(\frac{m\lambda}{2a}\right) = \sin^{-1}\left(\frac{1 \cdot 200}{2 \cdot 500}\right) = 11.54^\circ \)

(b) Normal incidence: \( \theta_b = \frac{1}{2} \sin^{-1}\left(\frac{m\lambda}{a}\right) = \frac{1}{2} \sin^{-1}\left(\frac{1 \cdot 200}{500}\right) = 11.79^\circ \)

5. (15%) P.13-5 of Pedrotti\textsuperscript{3}. Fresnel zone plate for a spherical light source.

The zone plate radii given by Eq. (13-20) were derived for the case of plane waves incident on the aperture. If instead the incident waves are spherical, from an axial point source at distance \( p \) from the aperture, show that the necessary modification yields \( R_p = \sqrt{nL\lambda} \), where \( q \) is the distance from aperture to the axial point of detection and \( L \) is defined by \( 1/L = 1/p + 1/q \).
See Figure 13-20. Let \( d_1 \) be the distance from the source point to the indicated point on the wavefront in the aperture, and \( d_2 \) be the distance from the indicated point to the observation point \( P \). For the \( n \)-th zone,

\[
(d_1 + d_2) - (p + q) = n\lambda / 2
\]

\[
d_1 = \sqrt{p^2 + R_n^2} = p\left(1 + \frac{R_n^2}{p^2}\right)^{1/2} \approx p\left(1 + \frac{1}{2}\frac{R_n^2}{p^2}\right)
\]

Likewise, \( d_2 \approx q\left(1 + \frac{1}{2}\frac{R_n^2}{q^2}\right) \)

\[
\therefore (d_1 + d_2) - (p + q) = \frac{1}{2}R_n^2\left(\frac{1}{p} + \frac{1}{q}\right) = n\lambda / 2
\]

\[
\rightarrow R_n = \sqrt{nL\lambda}, \quad \frac{1}{L} \equiv \frac{1}{p} + \frac{1}{q}
\]

6. (20%) Design a micromachined Fresnel zone plate.

You are asked to design a Fresnel zone plate using polysilicon (refractive index = 3.3) as the material. The zone plate is expected to work as a lens and focus a collimated laser beam (\( \lambda = 850 \) nm) to a primary focal point 0.5 mm away from the plate.

(a) What are the radii of the first three Fresnel zones in microns?

(b) Suppose we want to make this into a binary-phase Fresnel lens, what should be the thickness of the polysilicon plate, if we want this thickness to be as close to \( 2 \) \( \mu \)m as possible?

(a) \( R_N \approx \sqrt{Nf\lambda} \)

\[
R_1 = \sqrt{500 \cdot 0.85} = 20.6 \ \mu m
\]

\[
R_2 = \sqrt{2 \cdot 500 \cdot 0.85} = 29.2 \ \mu m
\]

\[
R_3 = \sqrt{3 \cdot 500 \cdot 0.85} = 35.7 \ \mu m
\]

(b) The polysilicon rings are supposed to cause \( \pi \) phase shift:

\[
\left(\frac{nt}{\lambda} - \frac{t}{\lambda}\right) \cdot 2\pi = q\pi, \ q = 1,3,5,...
\]

\[
\rightarrow t = \frac{q\lambda}{2n-1} = \frac{0.85}{2} \frac{1}{3.3-1} q = 0.185q \ (\mu m)
\]

Therefore, \( q = 11 \) gives \( t = 2.035 \) \( \mu m \), which is closet to \( 2 \) \( \mu m \).