Resonant Optical Cavities

6.1 GENERAL CAVITY CONCEPTS

Until now we have implied that the cavity is an integral part of any laser because of the feedback that it provides. This is indeed true, but there is more than just raw feedback involved. As we see in this chapter, the cavity provides the ultimate in frequency-determining properties of the laser. It is most important to understand the classical electromagnetic problem of a cavity so as to appreciate why a laser always oscillates at a cavity resonance, why the photon lifetime is so important to the threshold of oscillation condition, and why the cavity "filters" the spontaneous emission from the atoms in the cavity so that stimulated emission takes place to generate a coherent electromagnetic wave.

6.2 RESONANCE

Resonance of an electromagnetic wave at optical frequencies is no different than resonance of any other system, be it mechanical or electrical. There is always an interchange of energy in such a system between potential and kinetic forms in the case of a mechanical system, with attendant friction losses, or between electric and magnetic energy with resistive losses in an electromagnetic problem.
Quite often, the phenomenon of resonance gets lost in the mathematics when analyzing a low-frequency system; fortunately, a much simpler physical picture emerges when we consider systems where the wavelength is much less than the physical dimensions of the components.

To make the problem as familiar as possible, we consider the excitation of the cavity shown in Fig. 6.1 by an external source, such as a tunable laser or a variable-frequency oscillator. To keep the arithmetic to a minimum, we consider all waves, incident on the cavity from the left, inside the cavity, or transmitted through it to the right to be uniform plane waves of limited spatial extent transverse to the direction of propagation. Our task is to relate the fields, running wave intensities, and stored energy on the inside of the cavity to those quantities that we can measure on the outside.

Let us follow a wave as it bounces back and forth between the two mirrors. Consider the initial field at the plane just to the right of $M_1$, labeled by $E_0$. It propagates to $M_2$ and back to the starting plane and experiences an amplitude change of $\Gamma_1 \cdot \Gamma_2$ and a phase factor $\exp(-jk2d)$ as it travels that round trip and thus generates the field labeled $E^+_1$, which experiences the same trials and tribulations as $E_0$, and in turn generates $E^+_2$, and so on. At every point along the path from $M_1$ to $M_2$, the fields $E^+_1$, $E^+_2$, and so on are to be added to $E_0$ to which we assign the reference phase of $0^\circ$. This phasorial addition is shown in Fig. 6.2 where, because there is an assumed lagging phase angle, we have assumed that the round trip phase shift (RTPS), $2\theta = 2kd$, is almost but not quite an integral multiple of $2\pi$ radians. That deficiency is labeled by $\phi$ and is related to $kd$ by

$$2\theta = 2kd = q2\pi - \phi$$  \hspace{1cm} (6.2.1)  

where $q$ is now an integer (not the complex beam parameter). Note that if the angle $\phi$ is significant, the total field propagating to the right inside the cavity is merely the difference between the origin and the spiral of phasors, quite similar to the straight-line distance between the beginning and end of a coiled rope. Not much at all. But, if that rope is uncoiled, the distance becomes much larger.

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FIGURE 6.1. Optical cavity.
In a similar fashion, the total field $E_T$ will be many times the initial value $E_0$ if $\Gamma_{1,2}$ are close to 1 and $\phi = 0$. This point is most important and needs repeating many times over. The following quantities are all maximized by the simple equation $\phi = 0$: the total field traveling to the right (and thus to the left also); the magnetic fields associated with $E_v$; the intensities of the waves; the number of photons bouncing back and forth; and the stored energy. These physical facts are characteristic of resonance, which is defined by

$$\text{round trip phase shift} = (\text{RTPS}) = 2kd = q2\pi$$

(6.2.2)

This simple equation is most important and should be committed to memory for instant recall.

Equation (6.2.2) contains a lot of information that can be discussed in complementary ways. Since $k = \omega n / c = 2\pi / \lambda$, we can use one of the equalities to find the resonant wavelengths

$$k \cdot 2d = \frac{\omega n \cdot 2d}{c} = \frac{2\pi \cdot 2d}{\lambda} = q \cdot 2\pi$$

(6.2.3a)

or

$$d = \frac{q \cdot \lambda}{2}$$

(6.2.3b)

where $\lambda = \lambda_0 / n$. This view of resonance states that there has to be an integral number of half wavelengths between the two mirrors. This implies that the integer $q$ is a very large number for optical frequencies and reasonable size cavities as the following examples illustrate.

**Example 6.1** A semiconductor laser cavity

<table>
<thead>
<tr>
<th>Material</th>
<th>Gallium arsenide (GaAs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Index of refraction</td>
<td>3.6</td>
</tr>
<tr>
<td>Length of cavity $d$</td>
<td>100 $\mu$m = 0.01 cm = 3.94 $\times$ 10$^{-4}$ in.</td>
</tr>
</tbody>
</table>
Sec. 6.2  Resonance

Wavelength region of interest  

Thus

\[ q = \frac{nd}{(\lambda_0/2)} = \frac{3.6 \times 10^{-2}}{0.4 \times 10^{-4}} \text{ cm} = 900 \]

We can turn this problem around and ask, What is the wavelength for \( q = 899 \) and 901?

\[ \frac{\lambda_0}{2} = \frac{nd}{899} \quad \text{or} \quad \lambda_0 = 8008.898 \text{ Å for } q = 899 \]

\[ \frac{\lambda_0}{2} = \frac{nd}{901} \quad \text{or} \quad \lambda_0 = 7991.121 \text{ Å for } q = 901 \]

Example 6.2  A gas laser

<table>
<thead>
<tr>
<th>Material</th>
<th>Helium-neon gas at 1 torr</th>
</tr>
</thead>
<tbody>
<tr>
<td>Index of refraction</td>
<td>1.000+</td>
</tr>
<tr>
<td>Length of cavity</td>
<td>20 cm=8 in.</td>
</tr>
<tr>
<td>Wavelength region of interest</td>
<td>6328 Å</td>
</tr>
</tbody>
</table>

Thus \( q = 632, 111 \) for \( \lambda_0 = 6328.0025 \text{ Å} \)

and \( q = 632, 110 \) for \( \lambda_0 = 6328.0125 \text{ Å} \)

Note that there is only 0.010 Å difference between two adjacent wavelengths for the large gas laser, a difference beyond the resolution capabilities of most monochromators. Obviously, it was somewhat silly to carry out the calculation of wavelength to eight significant digits when the distance is only given to two. However, the point is that the wavelengths are very close together even for the small semiconductor and extremely so for a large gas laser.

Equation (6.2.2) can also be interpreted in terms of frequency \( v \).

\[ k \cdot 2d = \omega \cdot \frac{2nd}{c} = 2\pi v \cdot \frac{2nd}{c} = q(2\pi) \]  

\[ \nu = q \cdot \frac{c}{2nd} \]  

Because \( q \) is restricted to integer values, there are only discrete frequencies which obey the resonance condition. The separation between those frequencies is given by

\[ \nu_{q+1} - \nu_q = \frac{c}{2nd} \]  

For reasonable-size cavities, this separation yields numbers that should be comfortable to most. For the helium-neon laser cavity of example 2, this difference is 750 MHz. Although the frequency difference is important (it is called the free spectral range), do not lose sight of the resonant frequency given by \( v = q(c/2d) = 473.7553 \text{ THz for } \lambda_0 = 6328.002 \text{ Å} \), which is very high compared to those “comfortable” values.

Stimulated emission, which is the key issue with any laser, is always proportional to the energy \( (E^2) \); hence that stimulation will always be a maximum at a cavity resonance.
6.3 SHARPNESS OF RESONANCE: Q AND FINESSE

It should be obvious from Fig. 6.2 that \( \phi = 0 \) (or RTPS = \( q \cdot 2\pi \)) corresponds to the maximum internal field, but there would be a significant amplitude for a "small" deviation from the exact resonance. The issues to be addressed here are (1) how small, (2) what is the ratio of the fields at resonance to that at anti-resonance, and (3) what is the frequency selectivity of the cavity.

There are three interrelated characteristic parameters associated with a cavity that describe the resonance phenomenon: \( Q \) (quality), \( F \) (finesse), and \( \tau_p \) (photon lifetime). To derive an explicit relationship between the resonance, these quantities, and the construction of the cavity, we need an analytic description of the fields inside the cavity and their relationship to those exciting the cavity.

The total electric field on the right side of \( M_1 \) and traveling to the right (indicated by the superscript "+"), is given by the field \( E_0 \), which is that transmitted through the mirror from the source, plus the fields that have made \((1 \text{ to } N)\) round trips to \( M_2 \), back to \( M_1 \), and starting the \((2 \text{ to } N + 1)\) trip. The amplitudes of those fields are simply related to \( E_0 \) by the field reflection coefficient of each mirror. The phase of the \( N \)th component, \( E_N \), is delayed with respect to \( E_0 \) by \( N \) times the round trip phase shift of \( 2kd = 2\theta \) and \( N \) times the phase contributed by each mirror.* While this last contribution can be important, let us ignore it for now in the interest of simplicity.

\[
E^+_1 = \sum E^+_N = E_0 \frac{1}{1 - \Gamma_1 e^{-j2\theta}} + \ldots
\]

\[
= E_0 \left[ \frac{1}{1 - \Gamma_1 e^{-j2\theta}} \right] \quad (6.3.1)
\]

where \( \theta \) is the electrical (or optical) length of the cavity, and is equal to \( \omega d/c \).

The total field returning from \( M_2 \), traveling to the left (indicated by the superscript "-"), and incident on \( M_1 \) is just \( \Gamma_2 \) times \( E^+_1 \) times the round-trip phase factor.

\[
E^-_1 = \Gamma_2 e^{-j2\theta} E^+_1 = E_0 \left[ \frac{\Gamma_2 e^{-j2\theta}}{1 - \Gamma_1 e^{-j2\theta}} \right] \quad (6.3.2)
\]

Both (6.3.1) and (6.3.2) state (in mathematical terms) the conclusions of Fig. 6.2; namely, that the fields are a maximum when the denominators are a minimum (when \( 2\theta = q \cdot 2\pi \)).

The running wave intensities \( I^+ \) and \( I^- \) are simply related to \( E^{+,-} \) by \( I = E \cdot E^*/2\pi \) where the asterisk denotes complex conjugation. For the wave running to the right, we obtain

\[
I^+(z = 0^+) = \frac{|E_0|^2}{2\eta} \left[ \frac{1}{1 - \Gamma_1 \Gamma_2 e^{-j2\theta} - (\Gamma_1 \Gamma_2 e^{-j2\theta})^* + |\Gamma_1 \Gamma_2|^2} \right]
\]

\[
= I_0 \frac{1}{1 - 2|\Gamma_1 \Gamma_2| \cos 2\theta | + |\Gamma_1 \Gamma_2|^2}
\]

\[
I^+(z = 0^+) = I_0 \frac{1}{1 - 2|\Gamma_1 \Gamma_2|[1 - 2 \sin^2 \theta] + |\Gamma_1 \Gamma_2|^2}
\]

*The phase of the reflection coefficient is usually a slow function of frequency, and we neglect it.
\[ I^+(z = 0^+) = \frac{E_0^2}{2\eta} \left\{ \frac{1}{(1 - \sqrt{R_1 R_2})^2 + 4\sqrt{R_1 R_2} \sin^2 \theta} \right\} \]  

(6.3.3)

where we have assumed that the field reflection coefficients are real numbers and have substituted the power reflection coefficients \( R_{1,2} = |\Gamma_{1,2}|^2 \). The plane \( z = 0^+ \) is just to the right of the surface of \( M_1 \).

The quantity \( \frac{E_0^2}{2\eta} \) is an intensity and is simply the (power) transmission coefficient of \( M_1 \) times the incident intensity \( T_1 \cdot \left[ \frac{E_{\text{inc}}^2}{2\eta} \right] \), where \( T_1 = 1 - R_1 \) for lossless mirrors. In a similar fashion, the intensity transmitted through \( M_2 \) is the power transmission coefficient \( T_2 = 1 - R_2 \) times the intensity given by (6.3.3) since any thing that starts to the right from \( z = 0^+ \) impinges on \( M_2 \) at \( z = d \). After converting to reflectivities and incident intensity, we obtain an expression for the intensity or power transmission coefficient through the two mirrors.

\[
I_t = \left\{ \frac{E_0^2}{2\eta} = T_i \cdot I_{\text{inc}} = (1 - R_1) I_{\text{inc}} \right\} \left\{ \frac{T_2 = (1 - R_2)}{(1 - \sqrt{R_1 R_2})^2 + 4\sqrt{R_1 R_2} \sin^2 \theta} \right\}
\]

or

\[
T(\theta) = \frac{I_{\text{trans}}}{I_{\text{inc}}} = \left\{ \frac{(1 - R_1)(1 - R_2)}{(1 - \sqrt{R_1 R_2})^2 + 4\sqrt{R_1 R_2} \sin^2 \theta} \right\} \tag{6.3.4}
\]

The net transmission is obviously a maximum when the denominator is a minimum, but simple examples bring home the consequences more clearly.

**Example 1**

Assume identical mirrors \( R_1 = R_2 \) and let \( \sqrt{R_1 R_2} = R \). At resonance \( \sin^2 \theta = 0 \) and

\[
T(q\pi) = [(1 - R)^2/(1 - R^2)] = 1 \text{ (one)} \text{ independent of } R.
\]

Think about that for a moment. If each mirror’s reflectivity were 99%, then the wave encounters two such obstacles each with a transmission coefficient of 1%. Our first inclination would be to suggest that the overall transmission from left of \( M_1 \) to the right of \( M_2 \) would be \( 10^{-4} \). That is wrong because it ignores the coherence properties of the wave as was used in the derivation of (6.3.4), illustrating that we must first add the fields and then square to obtain the intensity.

There is one more startling fact that is contained in the statement that \( T(q\pi) = 1 \) independent of the mirror reflectivities. Since the transmission coefficient of \( M_2 \) is \( (1 - R_2) \), the value of \( I^+ \) must be \( 1/(1 - R_2) \) times the incident value so that the product would be \( I_{\text{inc}} \). For the case of the reflectivity being 99%, the intensity incident upon \( M_2 \) must be 100 times larger than the intensity used to excite the cavity.

The running wave intensities on the inside of the cavity can be much larger than those on the outside, including that used to excite the cavity.

We have not violated conservation of energy. Rather, this intensity enhancement is a manifestation of the energy storage capabilities of a cavity with highly reflecting mirrors. At antiresonance, \( \theta = (q + 1/2)\pi \), and the transmission coefficient is very small.

\[
T(\theta = (q + 1/2)\pi) = \frac{(1 - R)^2}{(1 + R)^2} = 2.53 \times 10^{-5}
\]
for $R = 0.99$. Thus the cavity expresses a loud and significant preference for those frequencies obeying the resonance condition.

A plot of the transmission coefficient given by (6.3.4) is shown in Fig. 6.3 where the horizontal axis is $[\theta/\pi - q]$. Thus, it can be changed by varying the frequency $\omega = 2\pi n$, the wavelength $\omega/c = 2\pi/\lambda_0$, a distance $d$, or an index $n$. While the vertical axis is plotted as a transmission coefficient, it can also serve as a measure of the relative fields, energy, or running intensities inside the cavity since $T_2$ is merely a sampling device for those quantities.

The quality factor ($Q$) of the cavity is a measure of the sharpness or selectivity of the resonance. If $v_0$ is the frequency of one of the peaks, then $Q$ is given by

$$Q = \frac{\nu_0}{\Delta v_{1/2}} = \frac{\omega_0}{\Delta \omega_{1/2}} = \frac{\lambda_0}{\Delta \lambda_{1/2}}$$

(6.3.5)

where $\Delta v_{1/2}$ (or $\Delta \lambda_{1/2}$) is the full width at half of the maximum (FWHM), and the horizontal axis is treated as a frequency axis for the frequency variables or wavelength (increasing in the opposite direction) for a $\lambda$ variation.

It is a rather tiring exercise in trigonometry of small angles to solve for the frequencies $v_{+-}$ which make the response decrease to 50% of the maximum to find an analytic expression for $Q$. For reasonable values of the product of the reflectivities, we can expand $\sin \theta$ around the peak:

$$\sin \theta_{+-} = \sin \left( \frac{(\omega_{+-})nd}{c} \right) = \pm \frac{1 - (R_1 R_2)^{1/2}}{(R_1 R_2)^{1/4}}$$

(6.3.5a)

FIGURE 6.3. The transmission through a Fabry-Perot cavity as a function of the electrical length measured in units of $\theta/\pi - q$. The three curves were plotted for $R_1 = R_2 = 0.9$, 0.8, and 0.7.
and thus

\[ \Delta v_{1/2} = \nu_+ - \nu_- = \frac{c}{2nd} \left( 1 - \left( R_1 R_2 \right)^{1/2} \right) \]

(6.3.6b)

Thus the cavity \( Q \) is given by \( q(c/2nd) \) divided by \( \Delta v_{1/2} \). We should recognize that \( q \) is merely the number of half wavelengths between the two mirrors; thus

\[ Q = \frac{q(c/2nd)}{\Delta v_{1/2}} = \frac{2\pi nd}{\lambda_0} \frac{(R_1 R_2)^{1/4}}{1 - (R_1 R_2)^{1/2}} \]

(6.3.7)

since

\[ q = \frac{nd}{\lambda_0/2} \]

Those who were introduced to \( Q \) at radio and/or microwave frequencies should be comfortable with the concept of \( Q \) as a measure of the sharpness of the resonance. Unfortunately, the numerical values of \( Q \) at optical frequencies are astronomical, primarily because of the smallness of \( \lambda_0 \). For instance, suppose that \( d = 1 \) m, \( R_1, R_2 = 0.99 \), and \( \lambda = 632.8 \) nm; then \( Q = 9.88 \times 10^8 \). But, of course, the resonant frequency is also astronomical: \( \nu = c/\lambda = 4.74 \times 10^{14} \). Only the half-width \( \Delta v_{1/2} \) is a familiar quantity, 480 kHz.

To avoid such large numbers and yet provide a measure of the filtering properties of the cavity, one uses another term, the Finesse \( (F) \).^4

\[ F = \frac{\text{free spectral range}}{\text{full width at half maximum}} = \frac{c/2nd}{\Delta v_{1/2}} \]

(6.3.8)

or

\[ F = \frac{\pi(R_1 R_2)^{1/4}}{1 - (R_1 R_2)^{1/2}} \]

For the numbers just used, this yields a more reasonable value of \( F = 313 \).

### 6.4 PHOTON LIFETIME

Closely related to the quality factor \( Q \) or the finesse \( F \) of a cavity is its photon lifetime. It is a time constant describing the build up or the decay of energy in a cavity and is one of the most useful (but simple) parameters describing a cavity. The steps used to derive an explicit formula are simple but very important because similar steps will be used at strategic points through the remainder of the text. The derivation will be the first instance of describing the physical phenomena by a rate equation (i.e., a differential equation in which the independent variable is time).

Consider Fig. 6.4, which depicts a simple cavity with a package of photons bouncing back and forth between the mirrors. Assume that at \( t = 0 \), there are \( N_p \) photons in this

^4Historically, the use of the term came from those using optical instruments, whereas the \( Q \) concept came from the lower-frequency domain. At optical frequencies, the mirror system is referred to as a Fabry-Perot cavity. It is quite often used as a bandpass filter to examine very narrow portions of a spectrum.